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MATH Expressions Common Core

Building a New Standard of Success

Houghton Mifflin Harcourt's

Math Expressions Common Core

A Research-based Approach

CONTENTS

OVERVIEW

Houghton Mifflin Harcourt's ***Math Expressions Common Core*** is a comprehensive, coherent, cumulative, rigorous, balanced, and research-based mathematics program for Grades K–6. At the heart of ***Math Expressions Common Core*** is the building of a math-talk community. Through their experiences in this rich math-talk community, students reach their learning destination—the ability to use formal math methods with understanding and with fluency. Built upon a foundation of mathematics education research and NSF-funded studies* and authored by a leader in the field of mathematics education, the program is proven to be effective in raising students' achievement.

The purpose of this document is to demonstrate clearly and explicitly the research upon which ***Math Expressions Common Core*** is based. This research report is organized by the major strands that guided development of the program:

- Focus and coherence, with meaningful progressions of learning across Grade levels
- Rigor, with high expectations for conceptual understanding and procedural fluency
- Habits of mind, with a focus on mathematical practices and problem solving
- Effective instruction through manipulatives, visual representations, and communication
- Assessment, with a focus on data-driven instruction and ongoing assessment
- Equity and access, to meet all students' needs through differentiation and intervention
- Technology, or the purposeful use of high-quality tools and technology to support mathematics teaching and learning

Each strand is supported by research. The content, activities, and strategies presented in ***Math Expressions Common Core*** reflect what we know about teaching for mathematical understanding and align to the Common Core State Standards for Mathematics.

To help readers make the connections between the research strands and the ***Math Expressions*** Common Core program, the following sections are used within each strand:

- Defining the Strand. This section summarizes the terminology and provides an overview of the research related to the strand.
- Research that Guided the Development of ***Math Expressions Common Core.*** This section identifies subtopics within each strand and provides excerpts from and summaries of relevant research on each subtopic.
- From Research to Practice. This section explains how the research data are exemplified in ***Math Expressions Common Core.***

A list of sources is provided at the end of this document.

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INTRODUCTION TO *MATH EXPRESSION COMMON CORE*

We live in a world that will place new challenges and demands upon its citizens. To be college- and career-ready, students will need to possess high levels of mathematical thinking and reasoning (National Council of Teachers of Mathematics, 2009). Fundamental changes in the economy mean that jobs will require higher levels of education. To compete, students will need to optimize their knowledge and problem-solving abilities (Partnership for 21st–Century Skills, 2008).

To fully participate in 21st–century society, students must build, at the earliest grades, the foundations for future learning. Researchers engaged in comparing U.S. performance with the performance of students worldwide, through the results of international assessments of mathematics including the Trends in International Mathematics and Science Study (TIMSS) and the Program for International Student Assessment (PISA), recommend that "countries that want to improve their mathematics performance should start by building a strong mathematics foundation in the early grades" (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005, p. v).

Before the development of the Common Core State Standards for Mathematics, critics of the American curriculum suggested it was too fragmented (with every state articulating a different set of standards and expectations) and too shallow (often referred to as "a mile high and an inch deep"). Students in the earliest grade levels were not developing essential foundations. Many teachers felt pushed to focus on building conceptual understanding—at the expense of procedural fluency—or on emphasizing rote memorization—at the expense of true understanding. Issues of equity resulting from inconsistent standards, curriculum, and assessments were a concern (Reed, 2009). Students learning in systems with different standards demonstrated wide disparities in performance on the National Assessment of Educational Progress (NAEP) (Schneider, 2007). In 2002, Schmidt, Houang, and Cogan concluded that the U.S. curriculum was "highly repetitive, unfocused, unchallenging, and incoherent" (p. 13).

Mathematics policy makers and researchers in the country saw a need to develop a world-class set of standards that could serve as a shared model for states. Mathematics educators saw the need for a curriculum that would offer a balanced program to meet students' needs.

The Common Core State Standards for Mathematics are focused, coherent, and rigorous—and describe the content and skills needed to "help students gain strong foundations, including a solid understanding of concepts, a high degree of procedural skill and fluency, and the ability to apply the math they know to solve problems inside and outside the classroom" (Common Core State Standards Initiative, 2015). The Common Core State Standards for Mathematics (CCSS-M):

- Are research-based
- Focus on critical skills at each grade level
- Encourage conceptual mastery of key ideas
- Develop students' mathematical understanding and procedural skill and fluency
- Build students' ability to apply math flexibly in context
- Present a coherent progression from grade to grade
- Prepare students for the demands of the future—in school and work

While the standards detail the knowledge and skills that students are expected to learn at each grade level, they do not describe the instructional approaches teachers will take to help students meet the standards. Thus, an effective instructional program is needed to bridge between the expectations set out by the standards and the desired student outcomes.

Houghton Mifflin Harcourt's ***Math Expressions Common Core*** was developed to respond to the same gaps and recommendations as the Common Core State Standards for Mathematics. ***Math Expressions*** is:

- **Focused** – Content is focused on essential learning, the core concepts of the Common Core State Standards for Mathematics, and big ideas at each grade so that students master content before they progress.
- **Coherent** – Content is organized into meaningful progressions that connect key topics between the grade levels.
- **Rigorous** – Content is presented for students to gain a deep conceptual understanding and facility with procedures used to solve problems.
- **Integrated** – The Standards for Mathematical Practice are incorporated into all of the lessons in ***Math Expressions.***
- **Balanced** – Students learning with ***Math Expressions*** will develop conceptual understanding and procedural fluency.

Math Expressions Common Core supports the Common Core State Standards for Mathematics by including work on all standards for each grade level but concentrating especially on the major work of the grade for deep mastery. The program's focus on visual representations, modeling, exploration and discussion offer research-based, engaging ways to teach and learn mathematics.

Research shows that the curriculum matters. According to Schmidt, Houang, and Cogan (2002) "One of the most important findings from TIMSS is that the differences in achievement from country to country are related to what is taught in different countries. In others words, this is not primarily a matter of demographic variables ... What we can see in TIMSS is that schooling makes a difference. Specifically, we can see that the curriculum itself—what is taught—makes a huge difference" (pp. 12–3).

Math Expressions is research-based, NSF-funded, and proven to raise achievement. The program was the subject of a curriculum study, examining the achievement effects of four different early elementary curricula for mathematic. The results? Researchers, Agodini, Harris, Thomas, Murphy, and Gallagher (2010), concluded that the "curriculum mattered" (p. 77). Students in Grades 1 and 2 using ***Math Expressions*** showed significantly higher achievement in mathematics than students using other programs with amount of teaching time controlled.

The ***Math Expressions*** commitment to presenting a research-based curriculum is essential today, in a landscape in which data-driven results and evidence-based teaching are the norm, rather than the exception. In their article, Clements and Sarama (2007b) set forth their position, "that education will not improve substantially without a systemwide commitment to research-based curriculum" (p. 137). ***Math Expressions*** matches this call: it is a program built upon decades of best-practice research in the mathematics classroom.

Together, the components of ***Math Expressions,*** K–6, offer a comprehensive, coherent approach to building students' foundational skills, conceptual understandings, and procedural fluency.

KEY RESEARCH IN THE DEVELOPMENT OF *MATH EXPRESSIONS*

The Author—Dr. Karen Fuson, Professor Emerita of Education and Psychology at Northwestern University and author of ***Math Expressions***, is a mathematics educator and developmental and cognitive scientist with decades of experience studying, researching, and writing about mathematics education.

Dr. Fuson's research focuses on how children learn math and the classroom conditions that support the development of students' understanding. Dr. Fuson's research for her *Children's Math Worlds (CMW)* NSF-funded project was instrumental in identifying key components for successful mathematics learning—building concepts, Math Talk, student leaders, quick practice, and building community. The body of research that forms the basis of ***Math Expressions*** focused on the following research tasks:

- Analyzing real-world mathematical situations to help curriculum developers and teachers select problems and examples that ensure both the understanding of the general math principles at work and of the real-world situation itself.
- Analyzing formal mathematical language and notation to identify difficulties that need to be addressed with pedagogical supports and classroom discussion.
- Developing meaningful real-world situations and visual supports that can facilitate interest and accessibility.
- Identifying meaningful language that can connect to the formal mathematical language (e.g., "break-apart partners" for addends, "unmultiplying" for dividing).
- Identifying typical student solution methods and learning paths through a domain to more-advanced solution methods.
- Developing accessible but mathematically-desirable algorithms that relate to common algorithms but that all students can understand and explain.
- Identifying typical student errors and how to overcome them.
- Choosing drawn quantity representation that can facilitate understanding of the domain situations or quantities.
- Monitoring grade-level placement of, and approaches to, important topics around the world.
- Writing teaching materials in a "learn while teaching" style that enables teachers to learn new ways of teaching and new solution methods.
- Developing classroom activity structures that can be used repeatedly with different math topics to cut down on classroom management issue.

From her research results, and collaborations with and knowledge of the research of others in the field, Dr. Fuson has designed effective teaching approaches and identified progressions of Pre-K to Grade 6 students' development/experiential understanding across different mathematical domains. (See a list of relevant research, organized by focus, at the end of this report, in the section titled "Project References and Additional Research Support for ***Math Expressions.***")

In addition to the research projects described above, Dr. Fuson served on the NRC committees (below) that summarized research and made recommendations. She then was on the feedback committee for the Common Core State Standards (CCSS), and was one of the authors of the CCSS progressions that expand upon and exemplify those standards.

The Houghton Mifflin Harcourt ***Math Expressions*** program is the result of this extensive research into how students learn math.

Key Research – As evidenced by the numerous studies and program references presented throughout this report, ***Math Expressions*** reflects what research shows about effective mathematics teaching and learning. The following publications present key findings foundational to the development of ***Math Expressions***.

> **Adding It Up: Helping Children Learn Mathematics** (National Research Council, 2001)
>
> Adding It Up presents a picture of mathematics learning from PreK to Grade 8. The Mathematics Learning Study Committee identifies five components of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) and presents research findings for how students develop this proficiency. The book provides key recommendations for specific changes and approaches in teaching, curricula, and teacher education that can improve students' mathematics learning.
>
> **How Students Learn: Mathematics in the Classroom** (National Research Council, 2005)
>
> This publication was written to build upon the earlier How People Learn, and tailor the findings in a more practical, useful way that teachers can immediately employ in their instructional practices. Full of detailed suggestions for research-based instructional activities, the book is designed to help teachers meet challenges and produce understanding, fluency, and problem-solving among their students.
>
> **Mathematics Learning in Early Childhood: Paths toward Excellence and Equity** (National Research Council, 2009)
>
> The result of a comprehensive review of the research on mathematics learning in early childhood, this publication identifies critical areas for early mathematics study that will enable all students to reach their potential in mathematics. The research reported suggests that improvements in early childhood mathematics education will also particularly support those students at-risk of falling behind in mathematics by providing them the strong foundations they need for future success.

ORIGINAL LEARNING VS REMEMBERING

Some math programs use what they call a spiral approach to math teaching. In the spiral approach a concept or math domain is introduced and then returned to repeatedly over the year and sometimes in the following year or years. In contrast the Common Core State Standards specify a focused and coherent set of standards at each grade level with the intent to focus deeply and thoroughly on the important things in each grade so that all students master the grade-level standards. The spiral approach confuses initial learning with retention/forgetting. Students require a deep extended learning period for the level of learning and understanding expected by the CCSS. Then to remember what they learned, they need to review and practice the content periodically. Teachers often say, "My students just don't remember what they learned last week!!!" But research shows that often many students did not actually learn the content last week, so of course they cannot remember what they did not learn.

There is a huge research literature establishing the importance of original learning on retention/forgetting and on how best to structure review once original learning has occurred (see, for example, Zechmeister and Nyberg, 1982, for an overview). Many research studies have contrasted “massed practice” and “distributed practice,” with distributed practice generally found to be better than massed practice for retention (e.g., Underwood, 1961; Cepeda et al., 2009). ***Math Expressions*** uses distributed practice in the Remembering pages, with practice closer together just after learning and then becoming spaced farther apart (Cepeda et al., 2006). This research about practice after original learning does not apply to the phase of initial learning of content, which needs to be extended and deep and not spiraled for complex math content.

In ***Math Expressions*** the phases of initial learning and later practicing to remember are clearly separate, and both are emphasized. Students spend extended time learning and discussing concepts in class, and they do homework about these concepts to deepen the original learning. Then after the unit is over, that content appears on the Remembering pages on and off throughout the year. Both the Unit Test and the Remembering pages allow the teacher to identify students who need additional focused learning time on particular content. This ideally comes outside of class right after the Unit Test.

Ten years of Massed Practice on Distributed Practice, B.J. Underwood, Psychological Review, 1961, 229-247.

Distributed practice in verbal recall tasks: A review and quantitative synthesis. Cepeda et al., Psychological Bulletin, 2006, 132, 354-380.

Optimizing Distributed Practice: Theoretical Analysis and Practical Implications, Cepeda et al, Experimental Psychology, 2009, 56, 236-246.

Zechmeister, Eugene B., and Nyberg, Stanley E., Human Memory: An Introduction to Research and Theory, 1982, Brooks/Cole, Monterey, CA.

A NOTE ON PRODUCTIVE DISPOSITION

The authors of *Adding It Up,* the publication on children’s mathematical learning by the National Research Council (2001), define productive disposition as “the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics” (p. 131). The sections of this report that follow describe research on conceptual understanding, procedural fluency, application, strategic competence, mathematical practices, and problem solving, and show how the research in these areas connects to the ***Math Expressions*** program. But students’ learning and developing skill in these areas is dependent on their belief that math is understandable and that, with effort, they are capable of learning math. This kind of a productive disposition is an important factor in students’ success.

How does one develop a productive disposition? Students develop a productive disposition as they engage in well-planned, purposeful learning activities; “Developing a productive disposition requires frequent opportunities to make sense of mathematics, to recognize the benefits of perseverance, and to experience the rewards of sense making in mathematics” (National Research Council, 2001, p. 131). Many aspects of the ***Math Expressions*** program support the crucial building of a productive disposition. The years of classroom research that underlie ***Math Expressions*** developed learning paths of supports and student strategies. These supports and strategies can move students from their initial knowledge to understanding of and fluency with formal mathematical methods and notation. ***Math Expressions*** builds a helping community within the classroom. It sets high-level mathematical goals for all students and concentrates on prerequisite competencies to bring all students to mastery.

For all major, grade-level topics, ***Math Expressions*** starts at the student's level and continually elicits their thinking, provides visual and linguistic supports to move them to understanding, and ends with extended fluency practice, while continuing the emphasis on understanding and explaining with Math Talk. The curriculum is organized into ambitious, core, grade-level topics, with structured supports to bring students to a higher mathematical level. Daily **Quick Practice** activities in the classroom provide opportunity and structure for developing student leadership and self-regulation. Together and individually, students build prerequisite skills and bring new skills to fluency. Eventually, all students take on leadership roles within the **Quick Practice** activities. Acting as a leader develops confidence in every student, regardless of achievement level. Through these roles, students gradually assume more responsibility for learning.

And this is a key aspect of ***Math Expressions***: everyone including the teachers is both a teacher and a learner. Students learn to be helpful, contributing members of a teaching-learning math community as they work and talk together. In such a learning environment, students are made to feel safe, trusted, and validated. In such classrooms, competence and confidence develop hand in hand, and all take the learning path together to mathematical proficiency.

STRAND 1: FOCUS AND COHERENCE

A standards-based curriculum combined with the creative use of classroom strategies can provide a learning environment that both honors the mathematical strengths of all learners and nurtures students where they are most challenged.

(McREL, 2010, p. 7)

Mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts. Therefore, the standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations build in previous years….Each standard is not a new event, but an extension of previous learning.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010b, online)

DEFINING THE STRAND

Standards offer benchmarks, goals, guideposts for teachers to ensure that they are helping students to build the foundations that they need to move onto the next grade level and to be ready for college and work. Standards can help to ensure that teachers are appropriately targeting instruction and can help students to set clear academic goals for learning.

In 2005, Lauer and colleagues sought to examine the assumption, made by the No Child Left Behind Act, that standards-based education leads to improved teaching and learning. These researchers conducted a synthesis of research looking at the connection between standards and education outcomes. Their findings? Standards appear to have "predominantly positive influences on student achievement, including that of at-risk students" (p. vi). Standards influence classroom content and instruction, and can lead to higher student achievement as teachers adopt new practices (Lauer, Snow, Martin-Glenn, VanBuhler, Soutemeyer, & Snow-Renner, 2005).

As discussed previously, in 2010, the Common Core State Standards for Mathematics (CCSS-M) helped to change the landscape of mathematics curricula in the United States with its focus on major concepts and big ideas at each grade. The CCSS-M were written with the goal of creating coherence—clearly linking and connecting concepts. The writers of the CCSS-M "drew on research on learning progressions" (Cobb & Jackson, 2011) during development. According to Schmidt, Wang, and McKnight (2005) coherence, or the pattern by which topics are introduced and build across grades, may be "one of the most critical, if not the single most important, defining elements of high-quality standards" (p. 554).

Math Expressions Common Core reflects this strong attention to focus and coherence. The program introduces content in carefully sequenced, focused progressions that align to the CCSS-M and to what researchers know about effective sequencing in math instruction.

RESEARCH THAT GUIDED THE DEVELOPMENT OF THE *MATH EXPRESSIONS* PROGRAM

Alignment to Standards

The description of standards or instruction as “a mile wide and an inch deep” has become a common way to describe expectations and instruction that cover many topics—but none to mastery. Past comparisons of mathematics curricula in the United States with the curricula of other countries suggested that the U.S. K through Grade 8 curriculum was “shallow, undemanding, and diffuse in content coverage” (National Research Council, 2001, p. 4).

In contrast, research suggests that a greater focus on fewer content areas can be more beneficial for students, leading to greater mastery.

In past international comparisons, American students were outperformed by students from other countries on assessments of math achievement (see TIMSS study by Gonzales, Williams, Jocelyn, Roey, Katsberg, & Brenwald, 2008, and PISA study by Baldi, Jin, Skemer, Green, & Herget, 2007). In an effort to unpack the specific factors contributing to this relatively low performance across grade levels, Ginsburg and colleagues concluded that “the distribution of [instructional] time across mathematics content areas differs in ways consistent with our findings about relative performance across content areas” (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005, p. v). For example, in comparing time spent on specific content areas, researchers found that “the United States devotes about half the time to its study of geometry—its weakest subject—that other countries spend” (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005, p. 22). International comparisons revealed that top-performing countries have curriculum in which they “present fewer topics at each grade level but in greater depth” (National Mathematics Advisory Panel, 2008, p. 20).

In other words, if teachers want to improve students’ performance across mathematical content areas, they would benefit from focusing instruction accordingly.

The Common Core State Standards for Mathematics (CCSS-M) were drafted in part as a response to these criticisms, and were written with the goal of defining deep and rigorous math education from Kindergarten through Grade 12.

In the CCSS-M, the major work at each grade level is the focus. The standards are focused and coherent, and they build upon the international comparisons referenced above as well as the research of content experts and the experience of members of the writing team. The standards integrate key ideas in what we know about mathematics education, including:

- A focus on properties, reasoning, and rigor
- A focus on procedural fluency
- A focus on using real-world situations and flexible solution methods

Across grades Kindergarten to Grade 5, the CCSS-M are built around five domains:

1. Operations and Algebraic Thinking
2. Number and Operations in Base Ten
3. Number and Operations-Fractions
4. Measurement and Data
5. Geometry

In Grade 6 geometry continues, and four new domains extend the earlier work: Ratios and Proportional Relationships, The Number System, Expressions and Equations, and Statistics and Probability. The content of the CCSS-M from Kindergarten through Grade 6 focuses on critical concepts and big ideas, and builds on students' foundations, preparing them to move onto more demanding math concepts, procedures, and applications.

Meaningful Progression across Grade Levels

In comparing math performance among students in the United States with the performance of students in higher-achieving countries, one repeated conclusion has been that "successful countries tend to select a few critical topics for each grade and then devote enough time to developing each topic for students to master it. Rather than returning to the same topics the following year, they select new, more advanced topics and develop those in depth" (National Research Council, 2001, p. 37). In contrast, as stated previously, the American curriculum has often been more diffuse and over-crowded in its content coverage, and lacked the focus of more effective curricula.

Because math learning occurs sequentially, building on previous learning and developing in sophistication, part of a discussion of content in mathematics must address the idea of sequence or progression. As stated previously, the coherence of standards, as illustrated by the logical progression across grade levels, is an essential element of effective standards. Researchers Cobb and Jackson (2011) reviewed the Common Core State Standards for Mathematics (CCSS-M) and concluded that the standards represent "a major advance in this regard" (p. 184) and that "the developers make good on their intention to focus on a small number of core mathematical ideas at each grade" (p. 184). The standards build on the foundations of earlier years, with new learning extending upon what has already been learned.

These kinds of strong learning progressions build deep content knowledge and build the complexity of student skills over time.

This focus on clear progression in the CCSS-M is intentional. After the publication of the CCSS-M, the Common Core Standards Writing Team (2013) convened again to revise and edit the clear progressions of skills and learning across grade levels, which had been their starting point when drafting the standards. According to the writing team, the CCSS-M "began with progressions: narrative documents describing the progression of a topic across a number of grade levels, informed both by educational research and the structure of mathematics" (p. 4). Once the standards were finalized, writers returned to refine the progressions and ensure alignment. In the current version of the progressions, "They note key connections among standards, point out cognitive difficulties and pedagogical solutions, and give more detail on particularly knotty areas of mathematics" (p. 4).

Examining the CCSS-M, one can see the clear attention to key concepts and big ideas across the grade levels and the focus on the major work at each grade level. At the earliest grade levels, students must develop the foundations that will allow them to study the more complex mathematical ideas that build on these foundations.

The most effective instructional programs will build on children's intuitive mathematical thinking and use that initial understanding to help children learn to solve problems, employ strategies, and engage in mathematical thinking (Carpenter, Fennema, Franke, Levi, & Empson, 2015).

In terms of content, research suggests that for the youngest children, developing a thorough understanding of number and of geometry and spatial measurement are developmentally appropriate and especially crucial to supporting later study (Cross, Woods, & Schweingruber, 2009). According to Cross and colleagues, "Developing an understanding of number, operations, and how to represent them is one of the major mathematical tasks for children during the early years" (p. 22). In addition, "Geometry and measurement provide additional, powerful systems for describing, representing, and understanding the world" (p. 35).

For young students, a deep understanding of number is essential. Students must develop an understanding of number that "includes understanding concepts of quantity and relative quantity, facility with counting, and the ability to carry out simple operations" (Cross, Woods, & Schweingruber, 2009, p. 22).

Also critical is an early understanding of geometry and measurement: "Geometry is the study of shapes and space, including two-dimensional (2-D) and three-dimensional (3-D) space. Measurement is about determining the size of shapes, objects, regions, quantities of stuff, or quantifying other attributes. Through their study of geometry and measurement, children can begin to develop ways to mentally structure the spaces and objects around them. In addition, these provide a context for children to further develop their ability to reason mathematically" (Cross, Woods, & Schweingruber, 2009, p. 35).

In the elementary grades, students must develop understanding and use of the big ideas in mathematics. "Mathematics learning in early childhood requires children to use several specific mathematical reasoning processes, also known as 'big ideas,' across domains. These big ideas are overarching concepts that connect multiple concepts, procedures, or problems within or across domains or topics and are a particularly important aspect of the process of forming connections" (Cross, Woods, & Schweingruber, 2009, p. 44).

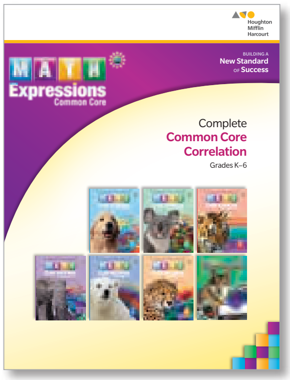
The "big ideas" are key concepts and procedures that can be used to teach varied math skills and processes. "Big ideas" in mathematics include concepts and procedures like the following:

- Place Value to One Million
- Addition with Multi-digit Numbers
- Subtraction with Multi-digit Numbers

Because these "big ideas" relate and connect to many other mathematical ideas, they help students to develop a deep understanding of mathematics as a set of ideas—not isolated facts or disconnected skills (Charles, 2005).

Worth noting is that not everything taught in mathematics fits neatly into a conceptual progression. While there is a temptation "to want to discover universal progressions in learning that are driven by deep changes in conceptual structure…there are parts of mathematics learning that, although important and complex, are driven by more incremental mechanisms" (Sherin & Fuson, 2005, p. 385). This does not suggest, however, that isolated, repeated practice is effective, but rather than there are some mathematical skills which may be best developed with practice in the context of a "meaningful examination of patterns and strategies" (p. 386).

FROM RESEARCH TO PRACTICE



Alignment to Standards in *Math Expressions*

Math Expressions Common Core was developed with the Common Core State Standards for Mathematics (CCSS-M), the Standards for Mathematical Practice, and the Learning Progressions as its foundation.

The program focuses on the priority core concepts at each grade level, identified by the Common Core State Standards for Mathematics, in order to build students' deep understanding of major mathematical ideas. The Standards for Mathematical Practice (discussed further in Strand 3 of this report) are incorporated into all lessons in the ***Math Expressions*** program.

This alignment between standards, curriculum, instruction, and assessments is critical. Researchers looking at effective educational practices identified nine characteristics of high-performing schools and reported that several of these relate to standards and standards alignment. High-performing schools have a clear, shared focus; high standards and expectations for all; and curriculum, instruction, and assessments aligned to the standards (Shannon & Bylsma, 2003).

The ***Math Expressions*** program reflects the key principles and emphases of the CCSS-M (as described in the table below), as well as aligning to the individual standards and the progression of the CCSS-M (as shown in the individual grade-level correlations provided online and within the Teacher's Edition).

Key Principles of the CCSS-M	How These Key Ideas Are Reflected in *Math Expressions*
Cumulative	• Progressions • Focus on Big Ideas • Opportunities for practice • Tools and resources for assessment
Balanced	• Focus on conceptual understanding and procedural fluency • Focus on content standards and mathematical practices
Research-Based	• Based on the NSF-funded Children's Math World project • Proven results demonstrated in large-scale study • Author papers series describes research on specific approaches • Evidence-based instructional approaches (manipulatives, representations, Math Talk community, and more)

Throughout ***Math Expressions,*** alignment with the Common Core is made explicit. In the Table of Contents, the Big Idea is followed by the correlating CCSS-M standards:

BIG IDEA 1 **Meanings of Multiplication and Division: 5s and 2s**

Common Core State Standards CC.3.OA.1, CC.3.OA.2, CC.3.OA.3, CC.3.OA.4, CC.3.OA.5, CC.3.OA.6, CC.3.OA.7, CC.3.OA.9

In the **Teacher's Edition,** units open with an Overview that shows which of the Common Core State Standards align with the lessons in the unit, as well as showing which standards are the prerequisites form the previous grades and which standards are the continuations in the next grade level.

Content Standards Across the Grades

Grade K	Grade 1	Grade 2
• Represent addition and subtraction and solve addition and subtraction story problems within 10. [CC.K.OA.1, CC.K.OA.2] • Decompose numbers less than or equal to 10 into pairs in more than one way. [CC.K.OA.3] • For any number from 1 to 9, find the number that makes 10 when added to the given number. [CC.K.OA.4] • Fluently add and subtract within 5. [CC.K.OA.4]	• Use addition and subtraction within 10 to solve story problems. [CC.1.OA.1] • Understand subtraction as an unknown partner situation. [CC.1.OA.4] • Relate counting to addition and subtraction. [CC.1.OA.5] • Demonstrate fluency for addition and subtraction within 10. [CC.1.OA.6] • Determine the unknown whole number in an addition or subtraction equation relating three whole numbers. [CC.1.OA.8]	• Use addition and subtraction within 100 to solve one- and two-step word problems. [CC.2.OA.1] • Fluently add and subtract within 20. [CC.2.OA.2]

Each unit begins with **Math Background** for the teacher to connect the topic of instruction with research and the Common Core State Standards for Mathematics.

The **Teacher's Edition** opens with a series of tables showing correlations between the ***Math Expressions*** program and the corresponding CCSS-M for that grade level.

Common Core State Standards for Mathematical Content

CC.2.OA Operations and Algebraic Thinking		
Represent and solve problems involving addition and subtraction.		
CC.2.OA.1	Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem.	Unit 1 Lessons 1, 2, 4, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21; Unit 2 Lessons 1, 2, 7, 15; Unit 4 Lessons 3, 4, 5, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23; Unit 5 Lessons 3, 4, 5, 6, 7, 9, 10; Unit 6 Lessons 8, 9, 14, 15; Unit 7 Lessons 3, 4, 5 **Daily Routine:** Money Routine
Add and subtract within 20.		
CC.2.OA.2	Fluently add and subtract within 20 using mental strategies. By end of Grade 2, know from memory all sums of two one-digit numbers.	Unit 1 Lessons 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21; Unit 2 Lessons 1, 2; Unit 3 Lessons 1, 2, 3, 4; Unit 4 Lesson 13; Unit 5 Lessons 3, 4, 5, 9, 10 **Quick Practices:** Unknown Addend; Stay or Go?; Equation Chains; Blue Math Mountain Cards; Dive the Deep; Make-a-Ten Cards: Addition; Make-a-Ten Cards: Subtraction; Addition Sprint; Subtraction Sprint ;Teen Addition Flash; Teen Subtraction Flash
Work with equal groups of objects to gain foundations for multiplication.		
CC.2.OA.3	Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.	Unit 1 Lessons 6, 7, 21; Unit 7 Lesson 1 **Quick Practices:** Count by 2s, Even or Odd?
	Use addition to find the total number of objects arranged in rectangular	

Meaningful Progression across Grade Levels in *Math Expressions*

Because she was a member of the Common Core State Standards for Mathematics (CCSS-M) feedback team, and participated in the creation of the CCSS-M progressions by the team in 2013, which are described as "an intermediate step between the Standards" and a standards-aligned textbook (p. 5), Dr. Fuson, author of ***Math Expressions,*** is uniquely positioned to convey the meaningful and clear CCSS-M progressions through a research-based, aligned curriculum.

Math Expressions and the Common Core State Standards for Mathematics share the same body of research as their foundation, the research summarized in the National Research Council Reports discussed above. Because of this shared research base, and because of author Dr. Karen Fuson's deep knowledge of effective, research-based math practices, the progression of teaching and learning in ***Math Expressions*** aligns precisely with the CCSS-M. In ***Math Expressions,*** mathematics content and models connect and build across the grade levels to provide a clear, meaningful, aligned progression of teaching and learning.

In ***Math Expressions,*** the sequence and progression of teaching and learning experiences is thoughtfully built upon a body of research on how young learners in mathematics develop number concepts and understanding of and skill with single- and multi-digit addition, subtraction, multiplication, and division; solving word problems; and fractions, ratio, and proportion. (See a list of this research, organized by focus, at the end of this report, in the section titled "Project References and Additional Research Support for ***Math Expressions.***")

In ***Math Expressions,*** ambitious, grade-level topics and big ideas are the center of the curriculum. Note the organization by Big Idea in the contents for Grade 3 on the next page.

To make the progressions clear to teachers, each unit opens with **Math Background,** which relates the lessons in the unit to the Learning Progressions for the Common Core State Standards, to show how the standards, and aligned lessons, build within and across grades.

In the **Teacher's Edition,** units open with an **Overview** that shows where the current lessons fall within the Learning Progressions for the Common Core State Standards:

Content Standards Across the Grades

Grade 2	Grade 3	Grade 4
• Represent and solve problems involving addition and subtraction. [CC.2.OA.1] • Add and subtract within 20. [CC.2.OA.2] • Work with equal groups to build foundation for multiplication. [CC.2.OA.3]	• Represent and solve problems involving multiplication and division. [CC.3.OA.1, CC.3.OA.2, CC.3.OA.3, CC.3.OA.4] • Apply properties of multiplication and the relationship between multiplication and division to multiply within 100. [CC.3.OA.5, CC.3.OA.6] • Fluently multiply and divide within 100. [CC.OA.7] • Solve problems involving the four operations, and identify and explain patterns in arithmetic. [CC.3.OA.9]	• Find factors and multiples of whole numbers. [CC.4.OA.4] • Analyze and generate patterns. [CC.4.OA.5] • Solve problems involving the four operations with whole numbers. [CC.4.OA.1, CC.4.OA.2, CC.4.OA.3]

Contents | Planning | Research & Math Background

UNIT 1 MULTIPLICATION AND DIVISION CONCEPTS FOR 0S, 1S, 2S, 3S, 4S, 5S, 9S, AND 10S.

REAL WORLD PROBLEM SOLVING

BIG IDEA 1 Meanings of Multiplication and Division: 5s and 2s

Common Core State Standards CC.3.OA.1, CC.3.OA.2, CC.3.OA.3, CC.3.OA.4, CC.3.OA.5, CC.3.OA.6, CC.3.OA.7, CC.3.OA.9

STRAND 2: RIGOR

Mathematics provides a powerful means for understanding and analyzing the world. Mathematical ways of describing and representing quantities, shapes, space, and patterns help to organize people' insights and ideas about the world in systematic ways. Some of these mathematical systems have become such a fundamental part of people's everyday lives—for example, counting systems and methods of measurement—that they may not recognize the complexity of the ideas underpinning them.

(Cross, Woods, & Schweingruber, 2009, p. 21)

Rigor refers to deep, authentic command of mathematical concepts, not making math harder or introducing topics at earlier grades. To help students meet the standards, educators will need to pursue with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skills and fluency, and application.

(National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010b, online)

DEFINING THE STRAND

To deeply understand mathematics, students must possess conceptual understanding, which enables them to make sense of mathematical problems and solutions, and procedural fluency, the skill which allows them to quickly, accurately, and flexibly solve problems. Successful application of mathematical ideas draws from both understanding and fluency.

Rigor in a set of curricular expectations is essential to students succeeding at high levels, but only if placed on a foundation of deep understanding and strong skills and fluency. While some have suggested that a solution to the problem of low student mathematical skills is to reduce the focus on computation and "simpler" math skills, research suggests that students' performance on items of low- and high-difficulty correlate highly—suggesting that students' "mathematical abilities to solve problems at different levels of mathematics rigor are complementary" (Ginsburg, Cooke, Leinwand, Noell, & Pollock, 2005, p. v). Deep understanding cannot be achieved without fluency and fluency cannot be reached without understanding.

The Common Core State Standards for Mathematics "promote rigor not simply by including advanced mathematical content, but by requiring a deep understanding of the content at each grade level, and providing sufficient focus to make that possible" (Achieve, 2010, p. 1).

Math Expressions Common Core continually enables students to work on and relate the three aspects of rigor in the Common Core State Standards for Mathematics: conceptual understanding, procedural skills and fluency, and application.

RESEARCH THAT GUIDED THE DEVELOPMENT OF THE *MATH EXPRESSIONS* PROGRAM

Conceptual Understanding

Conceptual understanding is the phrase used to describe one's "integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful. They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know" (National Research Council, 2001, p. 118). According to Cross and colleagues (2009), key here is both that students acquire knowledge and that they purposefully "access and apply this knowledge in new situations (Mayer, 2002)" (p. 244).

According to the research findings presented in *Adding It Up: Helping Children Learn Mathematics* (National Research Council, 2001), conceptual understanding benefits students because it allows them to learn more quickly because they are able to make connections between current knowledge and new topics. They can avoid critical errors because they can assess the reasonableness of solutions quickly.

In their study of mathematics learning in early childhood, Cross and colleagues (2009) concluded that to effectively foster students' conceptual understanding, teachers must include four key elements or opportunities within their teaching and learning activities:

1. Analyzing and reasoning
2. Creating
3. Integrating
4. Making real-world connections

In a study in which they compared students using a traditional control program with modified programs that employed worked examples, Booth and colleagues (2013) found that explaining worked examples—both correct and incorrect—during practice fostered deeper conceptual understanding.

In a report on two studies of young children from backgrounds of poverty, Fuson and Smith (2015) found that students were able to demonstrate high levels of conceptual understanding when they received instruction designed to provide opportunities to learn concepts and that employed strategies that included math drawings and intensive and conceptual learning experiences with visual supports.

Procedural Fluency

In its position statement on procedural fluency, the National Council of Teachers of Mathematics states that:

> Procedural fluency is a critical component of mathematical proficiency. Procedural fluency is the ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts; to build or modify procedures from other procedures; and to recognize when one strategy or procedure is more appropriate to apply than another.

All students need to have a deep and flexible knowledge of a variety of procedures, along with an ability to make critical judgments about which procedures or strategies are appropriate for use in particular situations for best success in the mathematics classroom (NRC, 2001, 2005, 2012; Star, 2005). The goal for students developing procedural fluency is that over time they will possess a body of known facts and generalizable methods that will allow them to efficiently and accurately solve varied problems.

A tension has existed historically in the United States between understanding and fluency. In describing a framework to describe effective teaching and learning in mathematics, Fuson (2009) describes this tension in more detail. She describes that some educators' misinterpretations of Piaget led to a greater emphasis on children's interactions with objects and activities—an emphasis on "understanding at the expense of fluency" which might have been "termed by its opponents as *mathematical learning without teaching*" (p. 345). In contrast to this approach is *mathematical teaching without learning,* in which rote practice and worksheets without attention to meaning-making are the focus. Fuson suggests that rather than having to choose one of these extremes, there is a balanced learning-teaching option, in between child-invented methods with too little structure and teachers' efficient methods with too much structure. In the balanced approach, teachers can structure learning paths for students based on their own progressions of learning, with a focus on both understanding and fluency. ***Math Expressions*** implements this balanced approach.

The findings of Ginsburg and colleagues (2005) support this balanced approach. While the Common Core State Standards for Mathematics were written to focus on specific areas of content, this was not accomplished by an arbitrary reduction of the focus on computation and skill building. Research has shown that skills with concepts and procedures are both essential to deep mathematical learning. An analysis of TIMSS and PISA results led researchers to conclude that "the evidence does not support proposals to reduce attention to learning computational and simpler mathematical skills in order to focus on strengthening students' ability to handle more complicated mathematics reasoning" (Ginsburg, Cooke, Leinwand, Noell, & Pollack, 2005, p. v). Instead, students need to focus each year on developing the skills that will allow them to perform well in low- and high-level problem-solving situations.

To achieve proficiency, students need instruction that recognizes the relationship between procedural fluency and conceptual understanding. Specifically, "Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems" (NCTM, 2014, p. 42). Effective mathematics instruction cannot have one without the other as "procedural knowledge and conceptual understandings must be closely linked" (NRC, 2005, p. 232). Rittle-Johnson and Alibali (1999) and Rittle-Johnson, Siegler, and Alibali (2001) found, too, that concepts and procedures develop iteratively—and gains in one area lead to gains in the other.

Research by Hiebert suggests that once students have memorized and practiced procedures that they do not understand, they have less motivation to understand their meaning or the reasoning behind them (Hiebert, 1999). When learning is not meaningful and is disconnected from other knowledge, students have a more difficult time absorbing concepts. When students are able to connect procedures and concepts, their retention improves and they are better able to apply what they know in different situations (Fuson, Kalchman, & Bransford, 2005).

Practice is key to developing procedural fluency. Research suggests that to be effective, teachers should create opportunities for practice that are brief, engaging, purposeful, and distributed over time (Rohrer, 2009). Worked examples, rather than additional practice problems, have also been shown to be effective in helping students learn to solve problems faster, perhaps because these worked problems help to reduce students cognitive loads and allow them to focus on the learning (Booth, Lange, Koedinger, & Newton, 2013). ***Math Expressions*** uses worked examples in the classroom by having students solve and explain methods to their classmates during the frequent Math Talk parts of a lesson.

In its position statement, NCTM concludes that to develop fluency students must have:

> …experience in integrating concepts and procedures and building on familiar procedures as they create their own informal strategies and procedures. Students need opportunities to justify both informal strategies and commonly used procedures mathematically, to support and justify their choices of appropriate procedures, and to strengthen their understanding and skill through distributed practice.

So, in sum, a wide body of research (see, for example, Baroody, 2006; Fuson & Beckmann, 2012/2013; Fuson, Kalchman, & Bransford, 2005; Fuson & Murata, 2007; Russell, 2000) suggests that to develop students' fluency in procedures, teachers should:

- Build on a foundation of conceptual understanding
- Support students in looking for patterns
- Allow students to flexibly choose among solution methods
- Offer distributed opportunities for purposeful, meaningful practice (not rote, repeated practice)

Practice occurs in ***Math Expressions*** lessons, on homework, and in the frequent distributed practice called Remembering.

Application

Rigor in the mathematics classroom can be seen as a three-legged stool, in which the three legs represent conceptual understanding, fluency, and application. An equitable, balanced attention of all three "legs" is essential to the effective teaching and learning of mathematics (Gaddy, Harmon, Barlow, Milligan, & Huang, 2014). Application draws on both conceptual understanding and procedural fluency, and also depends on and develops two of the five aspects emphasized in Adding It Up (2001): (1) strategic competence and (2) adaptive reasoning.

Application is emphasized in the CCSS-M and in ***Math Expressions.*** The CCSS-M "call for students to use math flexibly for applications in problem-solving contexts. In content areas outside of math, particularly science, students are given the opportunity to use math to make meaning of and access content" (Student Achievement Partners, 2012). The Operations and Algebraic Thinking (OA) domain of the CCSS-M emphasizes applications in the standards' specification of the various real-world problem situations that give meaning to the operations addition, subtraction, multiplication, and division.

Math Expressions has a powerful research-based approach to teaching such OA problem situations that stems from years of research. (See research on problem solving in the list of relevant research, organized by focus, at the end of this report, "Project References and Additional Research Support for ***Math Expressions.***") This approach begins with students making their own representation of the problem situation using a math drawing or a situation equation. As numbers get larger, students learn to represent problems with research-based diagrams. Student application of their conceptual understanding and procedural fluency in problem-solving situations is supported by the math talk in classrooms, during which students talk about aspects of how they apply their knowledge to problem situations. Students represent and solve all OA problem types for all quantities at the appropriate grade levels: single-digit numbers, multi-digit numbers, fractions, and decimal fractions. They also pose problems for classmates to solve at every grade level.

FROM RESEARCH TO PRACTICE

Conceptual Understanding in *Math Expressions*

Math Expressions systematically moves students through phases structured to build conceptual understanding, procedural fluency, and application.

Phase 1: Guided Introducing

In Phase 1, teachers elicit and the class works with the prior knowledge that students bring to a topic. Teachers and students discuss ideas and methods.

Phase 2: Learning Unfolding (Major Sense-Making Phase)

In Phase 2, teachers help students form conceptual networks and use methods that are desirable and accessible. Research-based solution methods from are discussed and explained. Math drawings and other supports help students correctly relate concepts and symbols and explain their thinking. Erroneous methods are analyzed and repaired, with explanation and discussion. Advantages and disadvantages of varied methods are discussed and compared.

Phase 3: Kneading Knowledge

Teachers help students gain fluency with desired methods. Students may choose a method, and can explain why it works. Some reflections and discussion still take place.

Phase 4: Maintaining Fluency and Relating to Later Topics

Teachers assist students in remembering by providing occasional problems and making explicit connections between new topics and prior knowledge.

In ***Math Expressions,*** specific features designed to help students avoid common errors help them to address misconceptions head on and develop concepts correctly. The Puzzled Penguin examples on page 24 show typical student errors—that students can then explain and teach correctly to Puzzled Penguin.

2-14 Class Activity

Name ______________________ Date __________

▶ **What's the Error?**

Dear Math Students,

My friends and I are helping build flower boxes for a community garden. We are going to build 42 flower boxes. The building plans say each box needs 13 nails. I rounded to estimate how many nails we'll need. Since 40 x 10 = 400, I bought a box of 400 nails.

My friends say we won't have enough nails. Did I make a mistake? Can you help me estimate how many nails we need?

Your friend,
Puzzled Penguin

12. Write a response to Puzzle Penguin.
Answers may vary. Responses should include that Puzzled Penguin used an underestimate. An overestimate is more appropriate in this situation.

Estimate and then solve. Explain whether the estimate is problematic in each situation. Answers may vary.

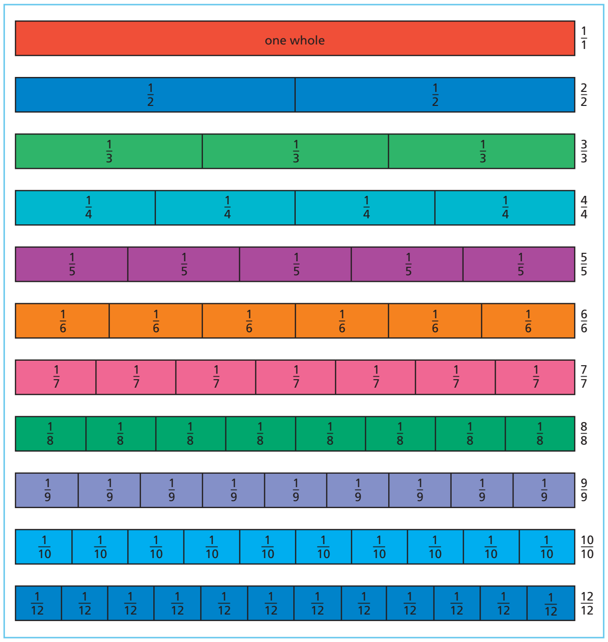
Math Expressions includes specific instructional activities designed to build students' conceptual understandings and address possible misconceptions.

<table>
<tr><th>Example of Students' Conceptual Difficulties</th><th>Example of How the Conceptual Difficulty Is Overcome in Math Expressions</th></tr>
<tr><td>Fractions: The fraction symbols and fraction words in English result in difficulties for students. The denominator of a fraction tells into how many equal parts the whole was divided. Thus, a larger number means a smaller unit. Furthermore, words that are used for order—third, fourth, fifth—are also used for fractions—leading to incorrect student generalizations for younger or struggling learners.</td><td>

• ***Math Expressions*** employs approaches and visual models designed to overcome these conceptual difficulties with fractions.

• **Quick Practice** activities offer additional opportunities for students to make connections and use important concepts.

• Drawings of fractions help students develop understanding.



</td></tr>
</table>

Additional tools that support students' development of conceptual understanding include **Math Mountains** and **Secret Code Cards,** which help students focus on the 10-ness of our number system as they learn to compose and decompose numbers and add, subtract, multiply, and divide to solve problems. These types of visual supports were developed through Dr. Fuson's research for the *Children's Math Worlds (CMW)* NSF-funded project.

Procedural Fluency in *Math Expressions*

Math Expressions includes a Fluency Plan for helping students achieve fluency with the Common Core State Standards for Mathematics at each grade, Kindergarten through Grade 6. This plan provides targeted practice in the Student Editions, Teacher Editions, Teacher's Resource Books, as well as Fluency Checks in the Assessment Guide.

The CCSS-M includes standards that set expectations for fluency at each grade and are reflected in the ***Math Expressions*** program. The chart on page 26 from the ***Math Expressions*** Teacher Edition highlights the Fluency Standards at each grade and the resources included in ***Math Expressions.***

Introduction | Pacing Guide | Contents | Common Core State Standards

Path to Fluency: Kindergarten through Grade 6

Math Expressions includes a Fluency Plan for helping students achieve fluency with the Common Core Standards that are suggested for each grade. This plan provides targeted practice in the Student Editions, Teacher Editions, Teacher's Resource Books, as well as Fluency Checks in the Assessment Guide.

Fluency and Memorization for Basic Facts

Kindergarten Fluency	Grade 1 Fluency	Grade 2 Memorization	Grade 3 Memorization	Grades 3, 4, 5, and 6 Intervention
K.OA.5 Fluently **add and subtract** within 5.	**1.OA.6 Add and subtract** within 20 demonstrating fluency for addition and subtraction within 10.	**2.OA.2** Fluently **add and subtract** within 20 using mental strategies. By end of Grade 2 know from memory all sums of two one-digit numbers.	**3.OA.7** Fluently **multiply and divide** within 100. By the end of Grade 3, **know from memory** all products of two one-digit numbers.	For those students who still need additional time for memorizing basic facts.
• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Fluency Checks (Assessment Guide)	• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Daily Routines (Teacher Edition) • Count-On Cards • Games • Fluency Checks (Assessment Guide)	• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Daily Routines (Teacher Edition) • Math Mountain Cards • Strategy Cards • Fluency Checks (Assessment Guide)	• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Practice Charts • Daily Study Plans • Study/Check Sheets • Dashes/Games • Strategy Cards • Diagnostic Tests • Fluency Checks (Assessment Guide)	Teacher's Resource Book: Grade 3: Addition and Subtraction Facts: • Diagnostic Quizzes • Practice Sheets Grade 4, 5, and 6: Multiplication and Division Facts: • Diagnostic Quizzes • Practice Sheets

Fluency for Operations with Multidigit Numbers

Grade 2 Fluency	Grade 3 Fluency	Grade 4 Fluency	Grade 5 Fluency	Grade 6 Fluency
2.NBT.5 Fluently **add and subtract** within 100.	**3.NBT.2** Fluently **add and subtract** within 1,000.	**4.NBT.4.** Fluently **add and subtract** multidigit whole numbers.	**5.NBT.5** Fluently **multiply** multidigit whole numbers.	**6.NS.3** Fluently divide multi-digit numbers using the standard algorithm **6.NS.4** Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation
• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Daily Routines (Teacher Edition) • Fluency Checks (Assessment Guide)	• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Fluency Checks (Assessment Guide)	• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Fluency Checks (Assessment Guide)	• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Fluency Checks (Assessment Guide)	• Path to Fluency Practice (Student Edition) • Quick Practices (Teacher Edition) • Fluency Checks (Assessment Guide)

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Quick Practice starts each lesson in ***Math Expressions.*** These practice activities help to develop fluency in core content.

With the program's **Accessible and Mathematically-Desirable Algorithms** students build on their own understandings to reach fluency.

Application in *Math Expressions*

Math Expressions has a powerful research-based approach to application—the third element of rigor. When students build conceptual understanding and procedural fluency, they must extend their new knowledge and skill into application.

The approach in ***Math Expressions*** begins with students making their own representation of a problem situation using a math drawing or a situation equation. As numbers get larger, students learn to represent problems with research-based diagrams.

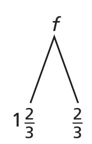
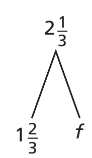
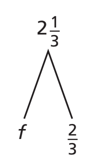
Student application of their conceptual understanding and procedural fluency in problem-solving situations is supported by **Math Talk,** during which students talk about aspects of how they apply their knowledge to problem situations.

Math Expressions provides many pathways to mathematical tasks. The program starts at the student's level and continually elicits thinking, provides visual and linguistic supports to move the student rapidly to understanding, and ends with extended fluency practice and application while continuing the emphasis on understanding and explaining.

Addition and Subtraction Problem Types

	Result Unknown	Change Unknown	Start Unknown
Add to	A glass contained $\frac{3}{4}$ cup of orange juice. Then $\frac{1}{4}$ cup of pineapple juice was added. How much juice is in the glass now? *Situation and solution equation:* [1] $\frac{3}{4} + \frac{1}{4} = c$	A glass contained $\frac{3}{4}$ cup of orange juice. Then some pineapple juice was added. Now the glass contains 1 cup of juice. How much pineapple juice was added? *Situation equation:* $\frac{3}{4} + c = 1$ *Solution equation:* $c = 1 - \frac{3}{4}$	A glass contained some orange juice. Then $\frac{1}{4}$ cup of pineapple juice was added. Now the glass contains 1 cup of juice. How much orange juice was in the glass to start? *Situation equation* $c + \frac{1}{4} = 1$ *Solution equation:* $c = 1 - \frac{1}{4}$
Take from	Micah had a ribbon $\frac{5}{6}$ yard long. He cut off a piece $\frac{1}{6}$ yard long. What is the length of the ribbon that is left? *Situation and solution equation:* $\frac{5}{6} - \frac{1}{6} = r$	Micah had a ribbon $\frac{5}{6}$ yard long. He cut off a piece. Now the ribbon is $\frac{4}{6}$ yard long. What is the length of the ribbon he cut off? *Situation equation:* $\frac{5}{6} - r = \frac{4}{6}$ *Solution equation:* $r = \frac{5}{6} - \frac{4}{6}$	Micah had a ribbon. He cut off a piece $\frac{1}{6}$ yard long. Now the ribbon is $\frac{4}{6}$ yard long. What was the length of the ribbon he started with? *Situation equation:* $r - \frac{1}{6} = \frac{4}{6}$ *Solution equation:* $r = \frac{4}{6} + \frac{1}{6}$

[1]A situation equation represents the structure (action) in the problem situation. A solution equation shows the operation used to find the answer.

	Total Unknown	Addend Unknown	Other Addend Unknown
Put Together/ Take Apart	A baker combines $1\frac{2}{3}$ cups of white flour and $\frac{2}{3}$ cup of wheat flour. How much flour is this altogether? *Math drawing:*[1]  *Situation and solution equation:* $1\frac{2}{3} + \frac{2}{3} = f$	Of the $2\frac{1}{3}$ cups of flour a baker uses, $1\frac{2}{3}$ cups are white flour. The rest is wheat flour. How much wheat flour does the baker use? *Math drawing:*  *Situation equation:* $2\frac{1}{3} = 1\frac{2}{3} + f$ *Solution equation:* $f = 2\frac{1}{3} - 1\frac{2}{3}$	A baker uses $2\frac{1}{3}$ cups of flour. Some is white flour and $\frac{2}{3}$ cup is wheat flour. How much white flour does the baker use? *Math drawing:*  *Situation equation* $2\frac{1}{3} = f + \frac{2}{3}$ *Solution equation:* $f = 2\frac{1}{3} - \frac{2}{3}$

[1]These math drawings are called math mountains in Grades 1–3 and break apart drawings in Grades 4 and 5.

STRAND 3: MATHEMATICAL HABITS OF MIND

If we really want to empower our students for life after school, we need to prepare them to be able to use, understand, control, modify, and make decisions about a class of technology that does not yet exist. That means we have to help them develop genuinely mathematical ways of thinking.

(Cross, Woods, & Schweingruber, 2009, p. 21)

DEFINING THE STRAND

What is mathematics? By looking at the many interrelated skills and knowledge involved in learning and doing mathematics, it is clear that mathematics is not simply a body of content or topics to be learned. Mathematics also encompasses ways of thinking and mathematical approaches that are essential to learning and doing math.

Learning in mathematics requires students to take a problem-solving approach, making connections and engaging in productive reasoning. Students of math must demonstrate persistence when initial methods or strategies do not generate solutions. Students who are successful in mathematics double check their solutions, to ensure that they have found a reasonable solution to the problem. All of these mathematical approaches can be taught and developed through modeling, practice, problem-solving opportunities—and an immersion in a classroom in which the Standards for Mathematical Practice are embedded within each lesson.

About twenty years ago, Cuoco, Goldenberg, and Mark (1996) proposed, in their classic article written with support from an NSF grant, that "more important than specific mathematical results are the habits of mind used by the people who create those results...this includes learning to recognize when problems or statements that purpose to be mathematical are, in truth, still quite ill-posed or fuzzy; becoming comfortable with and skilled at bringing mathematical meaning to problems and statements through definition, systematization, abstraction, or logical connection making; and seeking and developing new ways of describing situations" (p. 376). This suggestion—that a curriculum be organized around mathematical ways of thinking, or habits of mind—in some ways, with its focus on the how of learning instead of the what, anticipates the creation of the Common Core Standards for Mathematical Practice and the expression of these practices throughout ***Math Expressions***.

Math Expressions Common Core develops students' habits of mind, their mathematical practices, and their problem-solving approaches so that they are empowered to continue, in school and in life, with a questioning mind, making connections and approaching problems flexibly, thoughtfully, creatively, and persistently, with a goal of accuracy and clear communication of results. For example, teachers have reported that students voluntarily carry their high-level analyzing and explaining skills developed in math talk to non-math lessons.

RESEARCH THAT GUIDED THE DEVELOPMENT OF THE *MATH EXPRESSIONS* PROGRAM

Mathematical Practices

The Common Core Standards for Mathematical Practice

> …describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards … The second are the strands of mathematical proficiency specified in the National Research Council's report Adding It Up…(2010a)

In developing the *Principles and Standards* for School Mathematics, the National Council of Teachers of Mathematics (2009) identified expectations for content as well as for process. Under its Process Standards, NCTM includes Problem Solving, Reasoning and Proof, Communication, Connections, and Representation.

In attempting to define the many aspects of mathematics learning and understanding, the National Research Council (2001) identified five strands of mathematical proficiency:

> **Conceptual understanding**—Comprehension of mathematical concepts, operations, and relations.
>
> **Procedural fluency**—Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
>
> **Strategic competence**—Ability to formulate, represent, and solve mathematical problems.
>
> **Adaptive reasoning**—Capacity for logical thought, reflection, explanation, and justification.
>
> **Productive disposition**—Habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (p. 5).

The group concluded that "The integrated and balanced development of all five strands of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) should guide the teaching and learning of school mathematics" (National Research Council, 2001, p. 11).

The Common Core State Standards for Mathematics are an extension of these earlier efforts, by NCTM and the NRC, to define the processes and proficiencies of mathematics.

While the CCSS-M content standards indicate the concepts and skills students should learn, the Standards for Mathematical Practice describe how students should demonstrate their mathematical learning.

In the Common Core State Standards for Mathematics, the Standards for Mathematical Practice, "describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years" (NGA & CCSSO, 2010a). Students meet the Standards for Mathematical Practice by demonstrating the ability to:

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

The eight Common Core Mathematical Practices can also be organized into four broader categories, as shown here:

Math Sense-Making: **Make sense and use appropriate precision**	**Math Drawings:** **Model and use tools**
1. Make sense of problems and persevere in solving them 6. Attend to precision	4. Model with mathematics 5. Use appropriate tools strategically
Math Structure: **See structure and generalize**	**Math Explaining:** **Reason, explain, and question**
7. Look for and make use of structure 8. Look for and express regularity in repeated reasoning	2. Reason abstractly and quantitatively 3. Construct viable arguments and critique the reasoning of others

This organization permits a teacher to ask every day: "Today in my classroom did I do math sense-making about math structure using math drawings to support math explaining?" Then, the teacher can focus on becoming even better by asking: "And what more can I do tomorrow?"

Problem Solving

While some may see a dichotomy between gaining knowledge and applying knowledge, problem solving is a bridge between the two; solving problems enables students to build understandings while applying skills and knowledge. As Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier, and Wearne (1996) suggest, problematizing the subject links with the development of students' understanding; "Treating mathematics as problematic is the most powerful and practical way to think about problem solving" (p. 18).

The tasks with which teachers engage students in learning and doing mathematics is one of the most important instructional decisions that teachers make (Lappan & Briars, 1995). Tasks that allow students to make connections based on what they know, explore real-world problems, and promote higher-level thinking are particularly effective. The goal is that students "problematize with the goal of understanding the situations and developing solution methods that make sense" (Hiebert et al., 1996, p. 19).

Problem-solving situations are often placed into a meaningful or real-world context, and encourage students to make connections. These connections—among mathematical ideas, with other content areas, and in real-world contexts—are an essential part of mathematics learning. Making connections between new information and students' existing knowledge—knowledge of other content areas and of the real world—has proved to be more effective than learning facts in isolation (Beane, 1997; Bransford, Brown, & Cocking, 1999; Caine & Caine, 1994; Kovalik, 1994). Further, connecting mathematics to science, social studies, and business topics can increase students' understanding of and ability with mathematics (Russo, Hecht, Burghardt, Hacker, & Saxman, 2011). Students see the purpose and value of learning when they experience it in real-world contexts; "When instruction is anchored in the context of each learner's world, students are more likely to take ownership for...their own learning" (McREL, 2010, p. 7). According to Fosnot and Dolk (2010) teaching with contextual problems can be effective for developing "children's mathematical modeling of the real world" (p. 24). Connecting to the tasks improves their perception of the content as interesting and beneficial, thereby increasing their motivation to learn (Czerniak, Weber, Sandmann, & Ahem, 1999). Students learn best when what they learn seems relevant. [Although worth noting is Hiebert et al.'s (1996) argument that these real-life contexts can be engaging but are "not the primary determinant for engagement" (p. 18).]

Opportunities to solve problems helps students develop critical thinking skills. In a study that compared students exposed to teaching strategies that promoted higher-order thinking with those taught more traditionally, researchers found that experimental group students outperformed control group students, showing significant improvement in their critical-thinking skills; "Our findings suggest that if teachers purposefully and persistently practice higher order thinking strategies, for example dealing in class with real-world problems, encouraging open-ended class discussions, and fostering inquiry-oriented experiments, there is a good chance for a consequent development of critical thinking capabilities" (Miri, David, & Uri, 2007, p. 353). Student learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature (Boaler & Staples, 2008; Stein & Lane, 1996). And use of the mathematical practices as summarized above makes any task a high-level task that engages students in reasoning. The frequent use of math talk explaining in ***Math Expressions*** lessons lifts the students to engage in mathematical habits of mind.

Solving problems in the mathematics classroom has numerous benefits for students because they:

- Integrate their conceptual understandings with procedural fluency
- Develop more positive views of their abilities to solve problems
- Demonstrate and build persistence
- View the discipline of mathematics more positively

Key to these benefits is that students do the work needed to solve problems. When students solve problems which require them to choose and grapple with mathematical approaches, they engage in a productive struggle that is essential to learning mathematics and to developing the grit needed to persevere. Kapur (2010) found that students given time to make mistakes and persist through their struggles ultimately showed greater understanding on post-test measures than their counterparts. Engaging in problem solving teaches students how to employ strategies to solve problems—which helps them when they are faced with future problem situations (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier, & Wearne, 1996). Problematizing mathematics means to ask students to think for themselves and to explain their thinking, while supported by their teacher, classmates, and math program to struggle productively. Because the author of ***Math Expressions,*** Karen Fuson, was one of the authors of this 1996 paper, she understands this view deeply and implements it in ***Math Expressions.***

Effective scaffolds can be useful in a mathematics classroom focused on problem solving. As Hyde (2006) states, "Scaffolding does not necessarily make the problem easier, and the teacher does not do the work for students or show them how to do it. It enables the person to do it" (p. 28). Williams (2008) found that "scaffolding tasks allowed students to work independently at appropriately challenging levels, ...and develop a sense of self-confidence in their mathematics knowledge and skills" (p. 329). The research-based diagrams and math drawings used by students in ***Math Expressions*** lessons scaffold student thinking as they need it because they use these visual supports to scaffold their own thinking and explain to their classmates.

Ultimately, problem solving in the mathematics classroom encourages students to see that their actions can lead to intellectual growth, and this "focus on the potential of students to develop their intellectual capacity provides a host of motivational benefits" (Black, Trzesniewski, & Dweck, 2007, p. 260). This growth mindset is crucial in helping students think they can solve math problems.

FROM RESEARCH TO PRACTICE

Mathematical Practices in *Math Expressions*

Math Expressions has the Common Core Standards for Mathematical Practice as an extensive foundation. In ***Math Expressions,*** the Standards for Mathematical Practice are integrated directly into each lesson plan at the point of use. **Math Talk** is a key feature and an important vehicle to promote discussion that supports all eight of the Mathematical Practices. Each lesson includes a complete description of the activity and what teachers should expect from students, as well as explanations, sample questions, and student/teacher dialogs for **Math Talk.**

COMMON CORE
Mathematical Practice 3
Construct viable arguments and critique the reasoning of others.

Children use stated assumptions, definitions, and previously established results in constructing arguments. They are able to analyze situations and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others.

Children are also able to distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Children can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

MATH TALK is a conversation tool by which children formulate ideas and analyze responses, and engage in discourse. See also MP.6 Attend to Precision.

TEACHER EDITION: Examples from Unit 2

What's the Error? WHOLE CLASS

MP.3, MP.6 Construct Viable Arguments/Critique Reasoning of Others Puzzled Penguin Copy the addition example as shown below.

$$\begin{array}{r} 24 \\ 17 \\ 29 \\ +\ 26 \\ \hline 86 \end{array}$$

- Puzzled Penguin says this is one way to find the sum. What was Puzzled Penguin's mistake? Puzzled Penguin did not record the new tens correctly. 4 + 7 + 9 + 6 = 26, which has 2 tens. Puzzled Penguin only recorded one new ten.

Lesson 14 ACTIVITY 1

Is the Statement True?

Write the following on the board.

- A bag holds 4 nickels, 4 pennies, and 4 dimes. If 3 coins are taken out of the bag, the value of the 3 coins can never be more than 16 cents.

Children decide if the statement is true or false and use words and models to support their position. The statement is false.

MP.3 Construct Viable Arguments Justify Conclusions Children should be able to explain whether they think the statement is true or false and how they know they are correct.

Lesson 15 ACTIVITY 3

Mathematical Practice 3 is integrated into Unit 2 in the following ways:

Construct Viable Arguments	Puzzled Penguin
Critique the Reasoning of Others	Justify Conclusions
Compare Methods	

UNIT 2 | MATH BACKGROUND

UNIT 2 | Overview | 159U

Problem Solving in *Math Expressions*

Math Expressions takes a research-based problem-solving approach, in which students:

- Interpret the problem
- Represent the situation
- Solve the problem
- Check that the answer makes sense

This approach to problem solving incorporates all eight of the CCSS Standards for Mathematical Practice.

Mathematical Practice	Student Actions
Understand the Problem Situation **MP.1** Make sense of the problem. **MP.2** Reason abstractly and quantitatively.	**Make Sense of the Language** Students use the problem language to conceptualize the real world situation.
Represent the Problem Situation **MP.4** Model with mathematics. **MP.7** Look for and make use of structure.	**Mathematize the Situation** Students focus on the mathematical aspects of the situation and make a math drawing and/or write a situation equation to represent the relationship of the numbers in the problem.
Solve the Problem **MP.5** Use appropriate tools. MP.8 Use repeated reasoning.	**Find the Answer** Students use the math drawing and/or the situation equation to find the unknown.
Check That the Answer Makes Sense **MP.3** Critique the reasoning of others. **MP.6** Attend to precision.	**Check the Answer in the Context of the Problem** Students write the answer to the problem including a label. They explain and compare solutions with classmates.

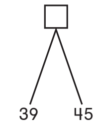
Math Expressions guides students through the processes and strategies they need to solve problems. **Break-Apart Drawings and Compare Bars** are examples of tools employed in ***Math Expressions*** to help students translate the words in a word problem into accurate situation equations.

Math Expressions includes all of the types of problems described in the CCSS-M and designed to build students' problem-solving ability and give them the opportunity to apply their conceptual understandings and procedural skills. This table shows the problem types with one unknown. The CCSS-M also includes Put Together/Take Apart problems with both addends unknown. ***Math Expressions*** also provides extensive experience with such problems beginning in Kindergarten.

Total Unknown

A clothing store has 39 shirts with short sleeves and 45 shirts with long sleeves. How many shirts does the store have in all?

Math Drawing[2]*:*



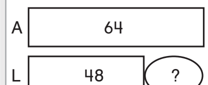
Situation and Solution Equation:
$39 + 45 = \square$

Difference Unknown

Alex has 64 trading cards. Lucy has 48 trading cards. How many **more** trading cards does **Alex** have than Lucy?

Lucy has 48 trading cards. Alex has 64 trading cards. How many **fewer** trading cards does **Lucy** have than Alex?

Math Drawing:



Situation Equation:
$48 + \square = 64$ or
$\square = 64 - 48$

Solution Equation:
$\square = 64 - 48$

Problem Types

This table shows how problem types are incorporated across the grades. A specific grade level problem types chart can be found at the back of each Student Book or Teacher Edition.

	Result Unknown	Change Unknown	Start Unknown
Add to	Six children were playing tag in the yard. Three more children came to play. How many children are playing in the yard now? *Situation and Solution Equation:* $6 + 3 = c$	Six children were playing tag in the yard. Some more children came to play. Now there are 9 children in the yard. How many children came to play? *Situation Equation:* $6 + c = 9$ *Solution Equation:* $9 - 6 = c$	Some children were playing tag in the yard. Three more children came to play. Now there are 9 children in the yard. How many children were in the yard at first? *Situation Equation:* $c + 3 = 9$ *Solution Equation:* $3 + c = 9$ or $9 - 3 = c$
Take from	Jake has 10 trading cards. He gave 3 to his brother. How many trading cards does he have left? *Situation and Solution Equation:* $10 - 3 = t$	Jake has 10 trading cards. He gave some to his brother. Now Jake has 7 trading cards left. How many cards did he give to his brother? *Situation Equation:* $10 - t = 7$ *Solution Equation:* $10 - 7 = t$	Jake has some trading cards. He gave 3 to his brother. Now Jake has 7 trading cards left. How many cards did he start with? *Situation Equation:* $t - 3 = 7$ *Solution Equation:* $7 + 3 = t$

	Total Unknown	Addend Unknown	Other Addend Unknown
Put Together/ Take Apart	Ana put 9 dimes and 4 nickels in her pocket. How many coins did she put in her pocket? *Situation and Solution Equation:* $9 + 4 = c$	Ana put 13 coins in her pocket. Nine coins are dimes and the rest are nickels. How many are nickels? *Situation Equation:* $13 = 9 + n$ *Solution Equation:* $13 - 9 = n$	Ana put 13 coins in her pocket. Some coins are dimes and 4 coins are nickels. How many coins are dimes? *Situation Equation:* $13 = d + 4$ *Solution Equation:* $13 - 4 = d$

	Difference Unknown	Bigger Unknown	Smaller Unknown
Compare[1]	Aki has 8 apples. Sofia has 14 apples. How many **more** apples does **Sofia** have than Aki? *Solution Equation:* $8 + a = 14$ or $14 - 8 = a$	**Leading Language** Aki has 8 apples. **Sofia** has **6 more** apples than Aki. How many apples does Sofia have? *Solution Equation:* $8 + 6 = a$	**Leading Language** Sofia has 14 apples. **Aki** has **6 fewer** apples than Sofia. How many apples does Aki have? *Solution Equation:* $14 - 6 = a$ or $6 + a = 14$
	Aki has 8 apples. Sofia has 14 apples. How many **fewer** apples does **Aki** have than Sofia? *Solution Equation:* $8 + a = 14$ or $14 - 8 = a$	**Misleading Language** Aki has 8 apples. **Aki** has **6 fewer** apples than Sofia. How many apples does Sofia have? *Solution Equation:* $8 + 6 = a$	**Misleading Language** Sofia has 14 apples. **Sofia** has **6 more** apples than Aki. How many apples does Aki have? *Solution Equation:* $14 - 6 = a$ or $6 + a = 14$

[1]The comparing sentence can always be said in two ways: One uses more, and the other uses fewer. Misleading language suggests the wrong operation. For example, it says *Aki has 6 fewer apples than Sofia*, but you have to add 6 to Aki's 8 apples to get 14 apples.

	Unknown Product	Group Size Unknown	Number of Groups Unknown
Equal Groups	Seth has 5 bags with 2 apples in each bag. How many apples does Seth have in all? *Solution Equation:* $5 \cdot 2 = n$	Seth has 5 bags with the same number of apples in each bag. He has 10 apples in all. How many apples are in each bag? *Situation Equation:* $5 \cdot n = 10$ *Solution Equation:* $10 \div 5 = n$	Seth has some bags of apples. Each bag has 2 apples in it. He has 10 apples in all. How many bags of apples does Seth have? *Situation Equation:* $n \cdot 2 = 10$ *Solution Equation:* $10 \div 2 = n$

	Unknown Product	Unknown Factor	Unknown Factor
Arrays[2]	Jenna has 2 rows of stamps with 5 stamps in each row. How many stamps does Jenna have in all? *Solution Equation:* $2 \cdot 5 = s$	Jenna has 2 rows of stamps with the same number of stamps in each row. She has 10 stamps in all. How many stamps are in each row? *Situation Equation:* $2 \cdot s = 10$ *Solution Equation:* $10 \div 2 = s$	Jenna has a certain number of rows of stamps. There are 5 stamps in each row. She has 10 stamps in all. How many rows of stamps does Jenna have? *Situation Equation:* $r \cdot 5 = 10$ *Solution Equation:* $10 \div 5 = r$
Area	The floor of the kitchen is 2 meters by 5 meters. What is the area of the floor? *Solution Equation:* $2 \cdot 5 = a$	The floor of the kitchen is 2 meters long. The area of the floor is 10 square meters. How wide is the floor? *Situation Equation:* $2 \cdot s = 10$ *Solution Equation:* $10 \div 2 = s$	The width of the kitchen is 5 meters long. The area of the floor is 10 square meters. What is the length of the floor? *Situation Equation:* $r \cdot 5 = 10$ *Solution Equation:* $10 \div 5 = r$
Compare	Katie picked 5 times as many flowers as Benardo. Benardo picked 2 flowers. How many flowers did Katie pick? *Solution Equation:* $5 \cdot 2 = k$	Katie picked 5 times as many flowers as Benardo. Katie picked 10 flowers. How many flowers did Bernardo pick? *Situation Equation:* $5 \cdot b = 10$ *Solution Equation:* $10 \div 5 = b$	Katie picked 10 flowers. Bernardo picked 2 flowers. How many times as many flowers did Katie pick as Bernardo? *Situation Equation:* $m \cdot 2 = 10$ *Solution Equation:* $10 \div 2 = m$

[2]Array problems can also be stated using the number of rows and columns in the array: The apples in the grocery window are in 3 rows and 6 columns. How many apples are there?

Note: All of the division situations could also have the multiplication equation as the solution equation because you can solve division by finding the unknown factor.

Students have ample opportunities for problem solving throughout ***Math Expressions.*** Real World Problem Solving is integrated throughout the lessons.

In multistep problems, students may need to find the answer to hidden questions needed to answer the question of the problem. ***Math Expressions*** guides students to see these hidden questions, even when they do not appear in the original problem.

The Teacher's Edition offers additional guidance in how to help students "Make sense of problems and persevere in solving them" (Common Core Mathematical Practice 1).

TEACHER EDITION: Examples from Unit 6

MP.1 Make Sense of Problems When adding mixed numbers, the wholes and the fractions are added separately. After the addition, the sum of the fractions might be a fraction greater than 1. This fraction then needs to be converted to a new mixed number. The additional whole is added to the other wholes.

Students can use Fraction Strips, draw fraction bars, or solve each problem numerically. If necessary, show students that they can add the wholes and fractions separately.

Lesson 5 ACTIVITY 2

MP.1, MP.4 Make Sense of Problems/ Model with Mathematics Draw a Diagram Problem 35 asks students to make a diagram to show that one of their solutions is correct. Students can make fraction bars or use another type of drawing that makes sense to them. Give students a couple of minutes to make their drawings and then choose volunteers to present and explain their work.

Lesson 6 ACTIVITY 2

Mathematical Practice 1 is integrated into Unit 6 in the following ways:

Make Sense of Problems	Draw a Diagram	Analyze the Problem
Look for a Pattern	Make a Graph	Make a Model

STRAND 4: EFFECTIVE INSTRUCTIONAL APPROACHES IN MATHEMATICS

Effective teaching is the non-negotiable core that ensures that all students learn mathematics at high levels.

An excellent mathematics program requires effective teaching that engages students in meaningful learning through individual and collaborative experiences that promote their ability to make sense of mathematical ideas and reason mathematically.

(NCTM Principles to Actions, 2014, pp. 4, 7)

Our examination of teaching focuses not just on what teachers do but also on the interactions among teachers and students around content. Rather than considering only the teacher and what the teacher does as a source of teaching and learning, we view the teaching and learning of mathematics as the product of interactions among the teacher, the students, and the mathematics...

(NRC, 2001, p. 313)

DEFINING THE STRAND

Teaching matters. The approaches that teachers take in the classroom can support students in learning and reaching their highest potential. A wide body of research has shown the impact of teacher effectiveness on student learning and achievement (Goldhaber, 2002; Partnership for Learning, 2010). Chetty, Friedman, and Rockoff (2012) looked at the long-term impacts of teachers and found that those who added value to their students' test scores also added life-long value to their students' educational attainment and income earning.

High-quality teachers use effective classroom practices (Wenglinsky, 2002). Research—in cognitive science, on classroom practices of master teachers, and on specific supports that help students learn—points to specific principles and methods of effective instruction (Rosenshine, 2012). Effective teachers engage students in deep learning. Teaching mathematics is not easy, but employing proven research approaches can help teachers ensure all students learn.

A wide body of research in mathematics supports the use of visual representations or drawings, created by teachers and students, and the incorporation of communication and classroom discourse as effective approaches for teaching mathematics. Visual representations can also be physical models or concrete objects that are used for teaching and learning (manipulatives), though drawings/diagrams have many advantages and are used extensively in ***Math Expressions***.

Both of these research-based approaches are evidenced throughout ***Math Expressions Common Core,*** in various activities and program features.

RESEARCH THAT GUIDED THE DEVELOPMENT OF THE *MATH EXPRESSIONS* PROGRAM

Visual Representations

According to the National Academies Press publication, *Mathematics Learning in Early Childhood,* "representing is central to mathematics" (Cross, Woods, & Schweingruber, 2009, p. 42). "Because of the abstract nature of mathematics, people have access to mathematical ideas only through the representations of those ideas" (National Research Council, 2001, p. 94).

According to NCTM, "Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; in communicating mathematical approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling" (NCTM, 2000, p. 67). Essentially, representations can show what students know, help students explain what they know, and be the foundation for making connections and achieving a deeper understanding of mathematics.

At every level, teachers and learners of mathematics use pictures or diagrams to represent situations. In mathematics, representations are not only written numbers or equations, the representations we immediately think of as "mathematical." In fact, representations include images, simple drawings, graphs, and other ways to see and think about mathematical ideas. Math drawings (or diagrams) are a part of the Common Core State Standards for Mathematics in all domains (OA, NBT, NF, MD, and G) from Kindergarten to Grade 6.

What these representations share is that they enable teachers to explain and learners to understand situations quantitatively or geometrically. Representations can help students to communicate, reason, problem solve, connect, and learn. Representations "help to portray, clarify, or extend a mathematical idea by focusing on its essential features" (NCTM, 2000, p. 206). Math drawings are tools for modeling, sense-making, reasoning, explain, structuring, and generalizing.

At the earliest grade levels, visual representations are particularly helpful in building students' understanding of number and geometry. Visual representations can help clarify concepts of tens and ones in the number systems—concepts that are made less clear by the structure of the English language. For young students, these visual representations and drawings of tens and ones can support understanding (Fuson, 2009). In a study with students in Grade 2, teachers successfully taught students to use schematic drawings to solve three-digit addition and subtraction word problems, and students demonstrated competence in choosing and applying the appropriate solution strategy (Fuson & Willis, 1989). Including visuals in the classroom can be particularly supportive of English learners and at-risk students (Fuson, Atler, Roedel, & Zaccariello, 2009; Fuson, Smith, & Lo Cicero, 1997).

Manipulatives are visual representations, as well; the term is used to refer to those concrete materials—such as blocks, cubes, base-ten blocks, place value cards (Secret Code cards), fraction strips, and so on—that teachers employ to develop students' mathematical understandings and skills. In addition to being manipulable and grounded in the concrete world, manipulatives also provide teachers and students with a visual point from which to have conversations about mathematical topics, concepts, and situations (Thompson & Lambdin, 1994). Research suggests that manipulatives can be effective in increasing students' mathematical knowledge (Clements & McMillen, 1996; Clements & Sarama, 2007b), particularly when care is taken in how students interact with the manipulatives. While manipulatives can be a primary vehicle for constructing knowledge, students will not automatically draw the same conclusions that their teachers draw; they must be helped to see the connections among the object, symbol, language, and concept (Ball, 1992a, 1992b).

In a study looking at Grades 1 and 2 students using concrete manipulatives to learn symbolic multi-digit addition and subtraction procedures, Fuson (1986) found that "for many children who made procedural errors on delayed tests, the mental representation of the procedure with the physical embodiment was strong enough for them to use it to self-correct their symbolic procedure" (p. 35). So, self-correction may be an additional benefit of manipulatives. This benefit also applies to the use of drawings.

Manipulatives are scaffolds for understanding mathematical concepts, notations, and vocabulary; they are a means and not an end (NRC, 2001). They need to be related to written methods to make those methods meaningful. After students master a concept using manipulatives and make the relationships with written methods, they can move to solving the task without the visual support (Grupe & Bray, 1999).

A recent study indicates that using visual representations has shown to improve student performance in general mathematics, prealgebra, word problems, and operations (Gersten, Beckmann, Clarke, Foegen, Marsh, Star, & Witzel, 2009). When students sketch or organize their mathematical thinking, they are able to explore their understanding of concepts, procedures, and processes—and communicate mathematically (Arcavi, 2003; Stylianou & Silver, 2004). Having students then participate in discussions about their representations allows for meaningful learning (Fuson & Murata, 2007). Visual representations are especially beneficial to students who have special needs, struggle with learning, or are English language learners, but they are necessary for all learners and teachers.

Communication

A wide body of research supports the important role of communication in the mathematics classroom. Communication is emphasized by the Common Core State Standards for Mathematics, which expect students to reason about math and discuss and explain their reasoning. Both National Research Council reports—*Adding It Up and How Students Learn*—emphasize discussion as a way to increase students' mathematical understanding. Discussions of cognitively challenging mathematical topics and ideas are a primary mechanism for promoting conceptual understanding (Michaels, O'Connor, & Resnick, 2008).

For younger children in the mathematics classroom, communication is also important in developing mathematical concepts and learning. As Cross and colleagues (2009) note, "The informal and formal representations and experiences need to be continually connected in a nurturing 'math talk' learning community, which provides opportunities for all children to talk about their mathematical thinking and produce and improve their use of mathematical and ordinary language" (p. 43) In the *Children's Math World (CMW)* studies, Fuson and colleagues conducted research on the crucial aspects of discussions and the best ways for teachers to transition to student-to-student discussion.

One phrase used to describe a classroom in which communication and discussion are primary vehicles for learning is the "Math-Talk Learning Community." Hufferd-Ackles, Fuson, and Sherin (2004, 2015) describe a math-talk learning community as one in which "individuals assist one another's learning of mathematics by engaging in meaningful mathematics discourse" (p. 81). In their research, they articulated the framework that enabled success for one teacher in an urban neighborhood in a class with English learners. They found that creation of a math-talk community requires teachers to move from the level of a traditional, teacher-directed classroom (Level 0 in their framework) to a classroom in which teachers coach and assist as students take leading roles (Level 3). To do so, students need to develop skills across the components of questioning, explaining mathematical thinking, identifying the source of mathematical ideas, taking responsibility for learning, and mathematical representations.

Discourse in the classroom connects to increased learning and achievement. Klibanoff and colleagues (2006) conducted a study of how the teachers' use of language impacted students' mathematical knowledge. To test this, the researchers transcribed teachers' language use, and found that the frequency of teachers' math talk correlated with students' increased mathematical knowledge.

A review of studies conducted by NCTM revealed that "the process of encouraging students to verbalize their thinking—by talking, writing, or drawing the steps they used in solving a problem—was consistently effective...Results of these students were quite impressive, with an average effect size of 0.98..." (Gersten & Clarke, 2007, p. 2). Hatano and Inagaki (1991) found that students who discussed and justified their solutions with peers demonstrated greater mathematical understanding than students who did not engage in such discussions. Leinwand and Fleischman (2004) reviewed research on effective math instruction and concluded that talking about math and explaining the rationale for solutions builds conceptual understanding.

Communication appears to help students persist in solving problems and to increase motivation and engagement. Research has also shown that learners are engaged in the learning process when they are asked to explain and reflect on their thinking processes (Good & Whang, 1999; Hettich, 1976; Surbeck, 1994). Continuing to work with students by asking them to explain and justify how they solved problems helps to maintain student engagement (National Research Council, 2001).

Teachers can engage in specific discourse practices to encourage students' "math talk." Asking "why?" and "how do you know?" is one strategy that effective teachers use to encourage students to explain their thinking, solve problems, and share mathematical strategies and ideas with their peers (Clements & Sarama, 2007, 2008; Thomson, Rowe, Underwood, & Peck, 2005). Instructional practices—such as restating, prompting students, and engaging in whole-class discussion, small-group discussion, and paired conversations—have been shown to be effective in improving student understanding (Chapin, O'Connor, & Canavan Anderson, 2003).

To foster a Math Talk Learning Community, teachers play an important role in engaging and involving students, managing discussions, and coaching students on productive, collaborative speaking and listening. First, teachers must model solutions and explanations. They must build listening skills, asking students to repeat in their own words. Teachers must demonstrate effective questions, asking for clarification and explanations. Only with these kinds of supports will students transition into effective student-on-student discussions.

To be effective, math discourse should:

- Build on students' thinking
- Provide ample opportunities for students to share ideas
- Engage students in analyzing and comparing approaches

The three phases of the teaching framework described by Fuson and Murata (2007) and summarized above emphasizes math discourse in phases 1 and 2 when conceptual understanding is being emphasized. Supports for such math talk are used throughout the ***Math Expressions*** program.

FROM RESEARCH TO PRACTICE

Visual Representations in *Math Expressions*

In ***Math Expressions,*** **Math Drawings** are a key part of learning.These drawings focus on the mathematical aspects of quantities or of a problem situation. In ***Math Expressions,*** both students and teachers use **Math Drawings** as tools for teaching and learning. These Meaningful Math Drawings are central to lessons in ***Math Expressions*** and are used together with **Math Talk** as students explain their thinking and listen to the explanations of other students. The use of math drawings enables students to work at their own entering level but move forward to build intertwined understanding and fluency.

As a tool for creating **Math Drawings,** ***Math Expressions*** uses individual dry-erase **MathBoards.** These boards can be used by students for representing and solving problems—and then displayed to share with peers and discuss the visual representations of problems and solutions. For example, see the **Research and Math Background** section of ***Math Expressions*** at Grade 2, Unit 2 for examples of how students use the **Mathboard** and **Secret Code** cards as tools to solve problems.

COMMON CORE

Mathematical Practice 5

Use appropriate tools strategically.

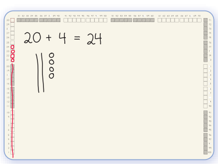
Children consider the available tools and models when solving mathematical problems. Children make sound decisions about when each of these tools might be helpful. These tools might include paper and pencil, a straightedge, a ruler, or the MathBoard. Children recognize both the insight to be gained from using the tool and the tool's limitations. When making mathematical models, they are able to identify quantities in a practical situation and represent relationships using modeling tools such as diagrams, grid paper, tables, graphs, and equations.

Modeling numbers in problems and in computations is a central focus in *Math Expressions* lessons. Children learn and develop models to solve numerical problems and to model problem situations. Children continually use both kinds of modeling throughout the program.

TEACHER EDITION: Examples from Unit 2

MP.5 Use Appropriate Tools MathBoard Modeling Children should place their MathBoards with the Number Path facing them or use the Number Path (TRB M34).

- How many tens and ones are in the number 24? 2 tens and 4 ones
- How can we show 24 on the Number Path? We draw Quick Tens through the first 20 squares, to show the two groups of 10. Then we draw circles on the next four squares to show the 4 ones.



Lesson 2 ACTIVITY 1

MP.5 Use Appropriate Tools Secret Code Cards Student Activity Book pages 105–106 give the directions for an activity that children can use to build fluency for addition within 100. Attach Demonstration Secret Code Cards to the board and invite two children to come to the front of the class to help you demonstrate how to do the activity. The focus of this activity is on deciding whether or not a new ten needs to be made.

Lesson 13 ACTIVITY 2

Mathematical Practice 5 is integrated into Unit 2 in the following ways:

Use Appropriate Tools	Secret Code Cards	Use Gestures
MathBoard	MathBoard Modeling	120 Poster

To make math concrete and meaningful to students, the activities in ***Math Expressions*** utilize visual representations, including manipulatives, for concept development. Student manipulatives include traditional ones, and unique research-based ***Math Expressions*** manipulatives like **Secret Code Cards** (for place value) and **Make a Ten and Product** cards (for meaningful practice) that provide students with visual models to promote understanding and procedural fluency. The strong emphasis in ***Math Expressions*** on representation and discussion open up the world of mathematics to all learners. There are opportunities for students to draw and view representations, listen to classmates discuss solution strategies and solutions, and explain and discuss their own strategies and solutions.

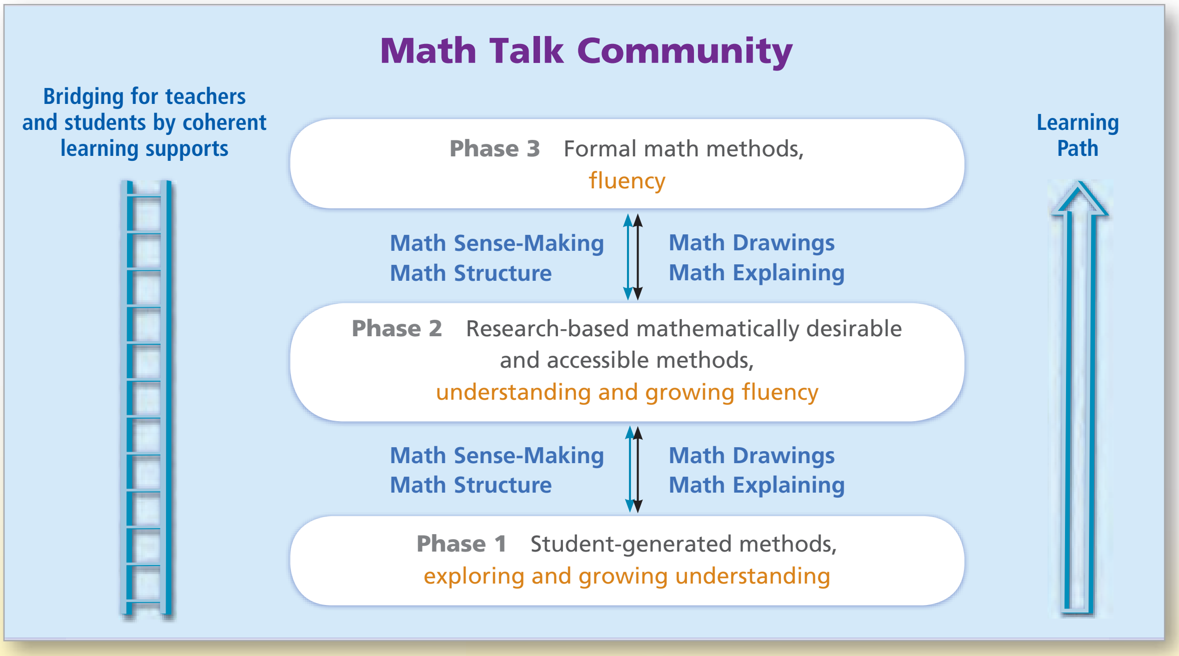
The program also includes ***iTools,*** electronic manipulatives that correspond to those used in the ***Math Expressions Common Core*** program. Online Interactive Whiteboard activities also provide an opportunity to use digital representations in the lessons.

Communication in *Math Expressions*

Math Expressions Common Core has at its heart the math-talk community. Within this student-centered community discourse is the shared way of building understandings and promoting one another's thinking and learning.

The opening of the ***Math Expressions*** **Teacher's Edition** includes background and professional learning for teachers about how to build a successful math-talk community.

In ***Math Expressions,*** each new topic begins with the teacher starting where students are and eliciting their thinking. As students continue with their study, they transition from the use of primitive solution methods to more formal methods. To reach the goal of fluent use of formal methods to solve mathematical problems, the program uses research-based approaches that have been shown to be effective and accessible to students. A focus on sense-making, structures, the use of drawing and representations, and the expectation that students will explain their choices and solutions contributes to student learning and progression.



In ***Math Expressions,*** the goal is for students to engage in student-to-student Math Talk. To engage students in this kind of talk, research demonstrate the effectiveness of asking students to:

- Solve
- Explain
- Question
- Justify

As described above, the program employs individual ***Math Expressions*** dry-erase **MathBoards.** These boards can be used by students for representing and solving problems—and then displayed to share with peers and explain the solution.

In ***Math Expressions,*** there are four key components of the **Math-Talk Learning Community.** Students and teachers work together, engaging in the kinds of roles and activities described in the table on the next page. The classroom works mostly at Levels 2 and 3.

This kind of **Math-Talk Learning Community** represents a shift from a traditionally organized classroom, in which the teacher is at the front of the room, delivering information and asking questions, with a focus on correctness, and students respond when asked for answers, to a more collaborative partnership for learning, in which students are engaged in an ongoing conversation for learning. Accuracy is just as important, but now accuracy can be judged by students using their understanding of concepts, methods, and problems instead of coming from the teacher without understanding.

A key structure of the Math-Talk Learning Community in ***Math Expressions*** is for students to **Solve and Discuss.** Selected students will go to the board, solve a problem, and then two or three of them explain their solutions to the whole group. Or, students can work in small groups where each student explains their method to the others in the group.

In ***Math Expressions,*** the teacher orchestrates collaborative instructional conversations focused on the mathematical thinking of classroom members. Together, students and the teacher use seven responsive means of assistance that facilitate learning and teaching by all (several may be used together).

Means of Assistance for Creating a Nurturing, Sense-Making, Math-Talk Community

- engaging and involving
- managing
- coaching: modeling, cognitive restructuring and clarifying, instructing and explaining, questioning, feedback

The teacher supports the sense-making of all classroom members by using and assisting students to use and relate:

- coherent mathematical situations
- pedagogical supports
- cultural mathematical symbols and language.

Levels and Components of a Math-Talk Learning Community				
	Questioning	**Explaining Mathematical Thinking**	**Mathematical Representations**	**Building Student Responsibility within the Community**
Level 0	Teacher is only questioner. Questions serve to keep students listening to teacher. Students give short answers and respond to teacher only.	Teacher question focus on correctness. Students provide short answer-focused responses. Teacher may tell answers.	Representations are missing or teacher shows them to students.	Culture supports students keeping ideas to themselves or just providing answers when asked.
Level 1	Teacher questions begin to focus on student thinking and less on answers. Only teacher asks questions.	Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in an explanation. Students provide brief descriptions of their thinking in response to teacher probing.	Students learn to create math drawings to depict their mathematical thinking.	Students feel their ideas are accepted by the classroom community. They begin to listen to each other supportively and to restate in their own words what another student said.
Level 2	Teacher asks probing questions and facilitates some student-to-student talk. Students ask questions of one another with prompting from teacher.	Teacher probes more deeply to learn about student thinking. Teacher elicits multiple strategies. Students respond to teacher probing and volunteer their thinking. Students begin to defend their answers.	Students label their math drawing so others are able to follow their mathematical thinking.	Students believe they are math learners and that their ideas and the ideas of classmates are important. They listen actively so that they can contribute significantly.
Level 3	Student-to-student talk is student initiated. Students ask questions and listen to responses. Many questions ask "why" and call for justification. Teacher questions may still guide discourse.	Teacher follows student explanations closely. Teacher asks students to contrast strategies. Students defend and justify their answers with little prompting from the teacher.	Students follow and help shape the descriptions of others' math thinking through math drawings and may suggest edits in others' math drawings.	Students believe they are math leaders and can help shape the thinking of others. They help shape others' math thinking in supportive, collegial ways and accept the same.

STRAND 5: ASSESSMENT

To be effective, intentional teaching requires that teachers use formative assessment to determine where children are in relation to the learning goal and to provide the right kind and amount of support for them to continue to make progress"

(Cross, Woods, & Schweingruber, 2009, p. 227)

Assessment…refers to all those activities undertaken by teachers—and by their students in assessing themselves—that provide information to be used as feedback to modify teaching and learning activities…

(Black & Wiliam, 1998a, p. 140)

We believe that assessment, whether externally mandated or developed by the teacher, should support the development of students' mathematical proficiency. It needs to provide opportunities for students to learn rather than taking time away from their learning.

(NRC, 2001, p. 423)

DEFINING THE STRAND

Assessment is an essential part of the effective instructional cycle (National Council of Teachers of Mathematics, 2000). Teachers rely on assessment data to provide diagnostic information on students' readiness. Ongoing, formative, informal and formal assessment data are essential for meeting the needs of all students, identifying when instruction has been successful and when additional support, intervention, or challenge opportunities are needed. Summative data provide essential benchmark data for results and future planning. As noted by numerous research studies, the regular use of assessment to monitor student progress can mitigate and prevent mathematical weaknesses and improve student learning (Clarke & Shinn, 2004; Fuchs, 2004; Lembke & Foegen, 2005; Skiba, Magnusson, Marston, & Erickson, 1986).

For students, too, assessment is key to learning. Research shows that the act of preparing for assessment and being assessed leads to greater learning. The feedback students receive from assessment helps them to evaluate their strengths and weaknesses, and gauge their progress towards meeting learning goals. Formative assessment has a positive effect on learning (Black & Wiliam, 1998b; Cotton, 1995; Jerald, 2001). In their research, White and Frederiksen (1998) found that low-achieving students had higher gains as a result of formative assessment, suggesting that effective assessment offers a tool for closing achievement gaps.

Math Expressions offers a comprehensive assessment program, with tools provided for each stage (diagnostic, formative, summative) and in varied formats. Throughout, teachers have the information they need to effectively plan and modify instruction to support all students to high levels of learning.

RESEARCH THAT GUIDED THE DEVELOPMENT OF THE *MATH EXPRESSIONS* PROGRAM

A Comprehensive Assessment Program

Essential to ongoing classroom instruction is the use of formative assessment. The phrase formative assessment encompasses the formal and informal tasks, conversations, activities, and observations that teachers employ regularly to measure student understanding and make and adjust instructional decisions. "Effective instruction depends on sound instructional decision-making, which in turn, depends on reliable data regarding students' strengths, weaknesses, and progress in learning content…" (National Institute for Literacy, 2007, p. 27). Effective teachers use formal (e.g., quizzes or homework) and informal tools (e.g., discussion or observation) to continuously monitor students' learning and progress (Cotton, 1995; Christenson, Ysseldyke, & Thurlow, 1989). Formative assessment is not an end; it is a means, to guide teaching and learning as it occurs (Shepard, 2000; Heritage, 2007). Formative assessment shifts assessment from being something "done *to* students; rather, it should also be done *for* students, to guide and enhance their learning" (NCTM, 2000, p. 22).

"Well-designed assessment can have tremendous impact on students' learning … if conducted regularly and used by teachers to alter and improve instruction" (National Research Council, 2007, p. 344). In its review of research, the National Mathematics Advisory Panel (2008) found that "use of formative assessments benefited students at all ability levels" (p. 46). A meta-analytic study by Baker, Gersten, and Lee (2002) found that achievement increased as a result of regular assessment use: "One consistent finding is that providing teachers and students with specific information on how each student is performing seems to enhance mathematics achievement consistently…The effect of such practice is substantial" (p. 67).

An additional benefit of formative assessment is that it has been shown to be particularly helpful to lower-performing students. Gersten and Clarke (2007) concluded that "the use of ongoing formative assessment data invariably improved mathematics achievement of students with mathematics disability" (p. 2). As a result, formative assessments can minimize achievement gaps while raising overall achievement (Black & Wiliam, 1998b).

Teachers must examine incorrect student responses to see if they "reveal specific student misunderstandings" (Popham, 2006, p. 86). By analyzing student errors, teachers can determine which specific concepts, algorithms, or procedures need additional instruction (Ketterlin-Geller & Yovanoff, 2009). Studying student responses as a group can also provide evidence of common misconceptions (NCTM, 2000).

Summative assessment also plays a role in the classroom. Checking student learning periodically in a unit and at the end of a unit offers insight when used as a point of information for subsequent instruction, as noted by Carnegie Mellon's Eberly Center Teaching Excellence and Educational Innovation (online). Summative assessments are also useful as accountability measures, for grading and gauging student learning against a set of standards or expectations. Summative assessments provide evaluative information to teachers about the effectiveness of their instructional program. Classroom summative assessments also appear to have an impact on student motivation, and have the potential to positively impact learning (Moss, 2013).

Varied Assessment Types and Options

Assessing students in meaningful ways is important to getting an accurate picture of students' progress and learning (Herman, Aschbacher, & Winters, 1992). For this reason, using varied items types and tasks may be the best way to get an accurate, complete reflection of student understanding; "Using multiple types of assessments provides more insight into students' learning because students have more than one way to demonstrate their knowledge and skills" (McREL, 2010, p. 44). Employing performance-based assessments can also help to assess multiple dimensions of learning (Marzano, Pickering, & McTighe, 1993).

As noted by Krebs' (2005) research, using one data point, such as written responses, to evaluate and assess students' learning can be "incomplete and incorrect conclusions might be drawn…" (p. 411). In defining the elements of an effective student assessment system, Darling-Hammond (2010) said that such a system must "address the depth and breadth of standards as well as all areas of the curriculum, not just those that are easy to measure" (p. 1). Variety in assessment types is an integral part of an effective comprehensive assessment program.

Asking students to respond to open-ended questions—in writing or through classroom discussion—is one useful way to assess what students are learning. As discussed by Moskal (2000) in her guidelines for teachers for analyzing student responses, students' responses to open-ended questions afford them the opportunity to show their approaches in solving problems and expressing mathematically what they know, which in turn allows the teacher to see the students' mathematical knowledge. Research by Aspinwall and Aspinwall (2003) on using open-writing prompts supports the use of open-ended questions in assessment in the mathematics classroom: "Students' responses to open-ended questions offer opportunities for understanding how students view mathematical topics… this type of writing allows teachers to explore the nature of students' understanding and to use this information in planning instruction" (p. 352-353). Similarly, by asking students to respond to open-ended questions verbally, researchers Gersten and Chard (2001) found that "encouraging students to verbalize their current understandings and providing feedback to the student increases learning." Researchers comparing student performance on assessments that include open-ended written responses with performance on multiple-choice tests found that students who wrote responses retained information better than those who responded to multiple-choice items (Roediger & Karpicke, 2006; McDaniel, Roediger, & McDermott, 2007). The ***Math Expressions*** Math Talk Community offers students continual opportunities to explain their thinking and relate their thinking to that of other students. This is a crucial aspect of the formative assessment in ***Math Expressions*** classrooms.

Multiple-choice items can play an important role in an assessment system as well. The National Mathematics Advisory Panel (2008) found that formative assessments based on items sampled from important state standards objectives resulted in "consistently positive and significant" effects on student achievement (p. 47). In addition, the Panel found multiple-choice items to be equally valuable in assessing students' knowledge of mathematics (National Mathematics Advisory Panel, 2008).

Economic and job trends internationally and technology innovations have necessitated that schools shift from fact-oriented curricula to emphasizing flexible, creative, effective approaches to problem solving (Fadel, Honey, & Pasnik, 2007; The School Redesign Network at Stanford University, 2008). In such an environment, performance-based assessment offer a needed tool that aligns with both how students learn and with the curricular emphases in the school.

Performance-based assessments connect to the important content and process skills emphasized in instruction, and offer students the chance to demonstrate their ability to classify, compare, analyze, or evaluate (Hibbard, 1996) and create a response or product. Performance-based tasks may take different forms, require different types of performances, and be used for different purposes (formative or summative) but they are typically couched in an authentic or real-life scenario and require high-level thinking. Darling-Hammond (2010) studied the characteristics of assessment systems in high-performing nations and found that "they emphasize deep knowledge of core concepts within and across the disciplines, problem solving, collaboration, analysis, synthesis, and critical thinking. As a large and increasing part of their examination systems, high-achieving nations use open-ended performance tasks …to give students opportunities to develop and demonstrate higher order thinking skills…" (p. 3)

In a standards-aligned system in which high-stakes assessments are a part of the landscape, different types of assessments are important. But the most fundamental aspects of assessment are those that the classroom teacher does to guide teaching. ***Math Expressions*** supports these formative assessments for teaching in various ways described below as well as supporting other types of assessments.

FROM RESEARCH TO PRACTICE

A Comprehensive Assessment Program in *Math Expressions*

Math Expressions provides a comprehensive assessment system, including:

- Diagnostic Tools
- Formative Assessment
- Summative Assessment
- Review Opportunities

See this overview of assessment resources from the Grade 3 Teacher's Edition for examples of the varied types of tools and resources provided in the program.

Assessment and Review Resources

DIAGNOSTIC TOOLS

Student Activity Book

- Unit Review and Test (pp. 293–294)

Assessment Guide

- Quick Quiz 1
- Quick Quiz 2
- Test A—Open Response
- Test B—Multiple Choice
- Performance Task

Online Test Generator

- Open Response Test
- Multiple Choice Test
- Test Bank Items

FORMATIVE ASSESSMENT

Teacher Edition

- Check Understanding (in every lesson)
- Quick Practice (in every lesson)
- Math Talk (in every lesson)
- Portfolio Suggestions (p. 649)

Assessment Guide

- Quick Quiz 1
- Quick Quiz 2

SUMMATIVE ASSESSMENT

Assessment Guide

- Test A—Open Response
- Test B—Multiple Choice
- Performance Task

REVIEW OPPORTUNITIES

Homework and Remembering

- Review of recently taught topics
- Spiral Review

Teacher Edition

- Unit Review and Test (pp. 647–650)

Assessment Guide

- Fluency Check (in every Big Idea beginning in Unit 3)

Online Test Generator

- Custom review sheets

In addition to Diagnostic Tools like those listed above, to diagnose at the beginning of the year, students can complete the Beginning of the Year Inventory Test (based on the prior year's standards).

Formative Assessment is a crucial piece of effective instruction, and is embedded within ***Math Expressions.***

As research shows, the frequent use of both formal and informal formative assessment is essential to effective, ongoing classroom instruction. ***Math Expressions*** provides many opportunities to regularly assess student understanding and make and adjust instructional decisions.

Formative Assessment Opportunities in *Math Expressions*

- **Check Understanding** appears at the end of each lesson in the **Teacher Edition.** This allows teachers to check students' understanding of the math content taught in the lesson.
- **Quick Quizzes** follow each **Big Idea** of the unit and are located in the **Assessment Guide.** These quizzes allow teachers to check students' understanding of the math content included in each **Big Idea.**
- **Math Talk and Math Drawing** opportunities are incorporated daily in the **Teacher Edition.** These opportunities for discussions allow teachers to see and hear the thought processes a student goes through when solving math problems. Teachers can intervene at the point of struggle, fix common errors, and remediate or accelerate learning for individuals or the whole class.
- **Math Writing Prompts** appear on the **Differentiated Instruction** page in the **Teacher Edition** for each lesson. There are three, leveled writing prompts that provide teachers an opportunity to check students' understanding of the math content through written communication.
- **Homework and Remembering** pages provide the teacher with daily feedback on student understanding.

The program also offers an ***Assessment Guide*** at each grade level with comprehensive assessment tools and resources.

While ***Math Expressions*** provides several tools and resources for ongoing formative assessment, perhaps as important are the daily insights gleaned by observation of students' work in the Math Talk Community. The Math Talk Community, supported by the program throughout the year, provides valuable data for formative assessment to guide teaching.

Finally, to keep all students on the grade-level learning path, ***Math Expressions*** is designed to support flexibility in offering more time and support to in-class periodic interventions and out-of-class Tier 2 and Tier 3 follow-up interventions. Teachers can use the **Mastery Learning Loop** to provide these kinds of periodic, in-class interventions for students who need the additional support. The **Mastery Learning Loop** is implemented as a full class period at specific times within the ***Math Expressions*** program pacing. In the **Mastery Learning Loop** for a given unit, a differentiation day occurs after each **Big Idea** and one or more such days occur at the end of each unit. In this way, the program helps teachers in assuring that all students master the content they need to move ahead.

Varied Assessment Types and Options in *Math Expressions*

Throughout ***Math Expressions,*** multiple effective types of assessment appear in order to best allow students to demonstrate their knowledge and skills.

Students complete open-response items to show their work and their processes. They engage in performance assessments that allow them to integrate multiple skills and demonstrate knowledge and skill in a problem-solving situation. The program provides ample opportunities for teachers to observe students at work, and make formative assessment decisions as a result.

Varied assessment types in ***Math Expressions*** include the following.

Varied Assessment Types and Options in *Math Expressions*

Assessment Type	Description and Examples
Multiple-Choice Assessments	Multiple-choice items allow for teachers to quickly get a sense of what students know and do not know, and can help to prepare students for on-demand, statewide assessments.
Mixed Response Formats	Mixed response format items—such as constructed-response items—allow for a deeper look at students' thinking and understanding of concepts and practices. Extended Response items allow a deeper view of student learning.
Performance Tasks	Performance tasks can reveal thinking strategies that students use to work through problems. A **Performance Assessment** activity for each unit provides the chance for students to apply understanding and skills—using logical reasoning, representing situations symbolically, using mathematical models to solve problems, and stating answers in terms of a problem context. Online Performance Tasks include a task for each unit with scoring rubrics and student work examples for scoring. ***Math Expressions*** also includes opportunities for **Portfolio Assessment.**
Common Core and High-Stakes Assessment Item Types	The HMH ***Getting Ready for PARCC®*, Smarter Balanced*, and High Stakes Assessment*** books provide students with exposure to and practice with the new item types that they may encounter on these assessments.
Online Assessments	The ***Math Expressions*** **Online Assessment** tools allow for immediate diagnosis of students' strengths and weaknesses, which can be followed with online activities and interventions. **Online Assessment** also provides students with experience taking tests in an online environment.

*This product is not endorsed by nor affiliated with PARCC or Smarter Balanced Assessment Consortium.

STRAND 6: MEETING THE NEEDS OF ALL STUDENTS

All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study—and support to learn—mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students.

(NCTM Principles & Standards for School Mathematics, 2000, p. 12)

Providing young children with extensive, high-quality early mathematics instruction can serve as a sound foundation for later learning in mathematics and contribute to addressing long-term systematic inequities in educational outcomes.

(Cross, Woods, & Schweingruber, 2009, p. 2)

DEFINING THE STRAND

Effective tools and strategies for differentiating instruction and offering intervention as needed is particularly crucial in today's diverse classrooms. In a single classroom, students may have diverse cultures, speak different languages, and differ in their prior knowledge, readiness, skills, motivations, interests, and learning styles (Tomlinson, 2005). These differences are important, because "Research has identified consistent, average differences in mathematics competence and performance depending on membership in a particular social group" (Cross, Woods, Schweingruber, 2009, p. 101). These differing needs must be effectively addressed. Teachers must help all students achieve because "All young Americans must learn to think mathematically, and…think mathematically to learn" (National Research Council, 2001, p. 1).

To help all students learn to think mathematically, teachers must meet them where they are. As Vygotsky (1978) noted in his seminal research on learning, "Optimal learning takes place within students' 'zones of proximal development'—when teachers assess students' current understanding and teach new concepts, skills, and strategies at an according level." Fuson and Murata (2007) describe a class learning path model that individualizes within whole-class activities by eliciting the whole range of student methods in Phase 1 of teaching a new topic and then moving in Phase 2 to ensure that mathematically powerful but accessible methods are introduced, discussed, and compared. This model enables all students to participate in the class discussion and understand at least one method and move on to a better method with support of the teacher and students. This approach reduces the need for differentiating in special groups.

Math Expressions has a strong emphasis on differentiation for teaching all learners and on intervention in response to specific needs. The program's emphasis on the visual (such as through **Math Drawings**) and the verbal (such as through **Math Talk**) provides a path to learning for all learners. ***Math Expressions*** uses the class learning path model and so initially does a great deal of differentiating in the whole class. It uses a **Mastery Learning Loop** model to target further specific differentiation. These approaches are discussed further below. ***Math Expressions*** addresses both of the quotes that began this section by providing extensive high-quality instruction followed by layers of differentiation to meet the needs of different students including high-achieving students.

RESEARCH THAT GUIDED THE DEVELOPMENT OF THE *MATH EXPRESSIONS* PROGRAM

Differentiation and Intervention

Students in the classroom vary in important ways. Tomlinson (1997) puts it plainly, "Students are not all alike. They differ in readiness, interest, and learning profile…" (p. 1). Teachers today face the challenge of meeting the needs of this increasingly diverse student population.

Effective instructional approaches can meet the learning needs of all students regardless of their background. According to recent findings by Fuson and Smith (2015), children from poverty can develop high levels of conceptual understanding—so long as they are taught using instructional approaches that support the learning of concepts. When kindergarten, first, and second graders from varied backgrounds had extensive opportunities to learn decomposing numbers and make drawings to solve addition and subtraction problem, they demonstrated high levels of performance. According to the researchers, "It is time to stop doing studies whose goal is to show that U.S. children or children from poverty do not understand math concepts. Now is the time for a substantive national discussion of instructional approaches that support the learning of concepts" (p. 42).

Talking about math has been found to benefit students at different levels of learning and in different contexts. In their study, Hufferd-Ackles, Fuson, and Sherin (2004) found a math-talk community to be beneficial with students who were English language learners in an urban setting. Similarly, working in a transitional language classroom led researchers Bray, Dixon, and Martinez (2006) to concluded that as students "communicate verbally and in writing about their mathematical ideas, they not only reflect on and clarify those ideas but also begin to become a community of learners" (p. 138).

Many years of research in classrooms provide the foundation for the ***Math Expressions*** program. This research produced learning paths for many topic areas that enable all children to learn. Knowledge of typical errors enables teachers to uncover and correct these quickly. The use of the class learning path model with the teaching phases and **Math Talk Learning Community** enable the whole learning path to be active within the whole-class discussions and thus to differentiate within these whole-class discussions. The **Mastery Learning Loop** specifies that teachers keep teaching the lessons in each **Big Idea** (a part of a unit) and then have an intervention day. Another intervention day can occur if needed before the unit test. This enables many strugglers to catch up and stay on level. Students who are way behind or have special learning difficulties may require further special intervention. High-achieving and on-level students receive differentiated learning activities on the intervention day, so that everyone gets some differentiation that follows initial high-quality instruction using the approaches summarized above.

This approach is the same as described in the recent research on Response to Intervention (RtI) models. These models frame differentiation as a prevention system with multiple layers—a structured way to prevent struggling students from falling behind—and so it focuses on early, and ongoing, identification of needs and tiers of responses. RtI integrates instruction, intervention, and assessment with the goal of increased student achievement (Mellard & Johnson, 2008).

Most commonly, RtI is implemented as a three-tier model where Tier 1 represents general instruction and constitutes primary prevention. Students at this level respond well to the general curriculum and learn reasonably well without additional support. Tier 2 represents a level of intervention for students who are at risk. Students at Tier 2 receive some supplementary support, in the form of instruction or assessment. Tier 3 typically represents students who need more extensive and specialized intervention or special education services (Smith & Johnson, 2011).

A number of studies attest to the effectiveness of this kind of intervention approach. Ketterlin-Geller, Chard, and Fien (2008) saw improvement in mathematics performance on various achievement measures when underperforming students were given structured intervention support. Fuchs, Fuchs, and Hollenbeck (2007) looked at RtI in mathematics with students in Grade 1 (a comprehensive program) and Grade 3 (a focus on word problems). They found that the data supported RtI at both grade levels, and showed "how two tiers of intervention, designed strategically to work in supplementary and coordinated fashion, may operate synergistically to decrease math problem-solving difficulties for children who are otherwise at risk for poor outcomes" (p. 19).

A publication from the What Works Clearinghouse of the U.S. Department of Education (Gersten, Beckmann, Clarke, Foegen, Marsh, Star, & Witzel, 2009) presents an overview of research and best practice in RtI for the elementary and middle grades. At Tier 1, they recommend screening to identify those at risk. At Tiers 2 and 3, they recommend that:

- Grades K through 5 focus on whole numbers; Grades 4 through 8 on rational numbers.
- Intervention instruction is explicit and systematic.
- Intervention in mathematics should include:
  - Models of problem solving
  - Word problems
  - Graphic organizers
  - Visual representations
  - Practice for fluency
  - Communication about math
- Students' progress should be monitored, and include practice, feedback, and review.
- Interventions should be designed to motivate students. (Gersten, Beckmann, Clarke, Foegen, Marsh, Star, & Witzel, 2009)

Math Expressions has all of these characteristics in its approach to differentiation. Screening for Tier 1 is accomplished by the continual formative assessment occurring in the **Math Talk** and other parts of lessons, in quizzes, and in unit tests. The differentiation for all Tiers and for on-level and advanced students occurs during the differentiation days as discussed in the **Mastery Learning Loop.** Tier 3 students receive further differentiation as needed.

FROM RESEARCH TO PRACTICE

Differentiation and Intervention in *Math Expressions*

With the program's focus on visual learning, mathematical representations, and math-talk discussion, ***Math Expressions*** opens the world of mathematics to all learners, whatever their learning style. Math drawings can help all students see concepts and understand solutions. The program's **Math Talk Learning Community** approach is especially valuable for English learners and native speakers who need additional practice developing their verbal skills. These visual and verbal pathways allow students to use their strengths—visual-spatial or language—to shore up their weaker pathways and build a connected web of understanding.

The *Children's Math World (CMW)* NSF-funded research upon which ***Math Expressions*** is based was conducted first in primarily urban, English- and Spanish-speaking classrooms and then later expanded into suburban schools with students of varied cultural backgrounds and diverse socio-economic status. The **Math Drawings** and the use of **Math Boards** and **Math Talk** in the program are part of a teaching-learning approach designed to be particularly effective with students who have been historically underrepresented in math and science occupations and to result in equal opportunities to learn in the mathematics classroom.

The many informal classroom research projects at that time and two studies of Kindergarten and Grade 1 children of poverty on their performance on the kinds of word problems suggested by the Common Core State Standards (Fuson and Smith, 2015) found that students taught using ***Math Expressions,*** with an emphasis on math drawings and an opportunity to learn concepts, were able to reach high levels of performance with concepts. Other papers summarize instructional approaches that are central aspects in the ***Math Expressions*** classroom and were powerful for children of poverty and also for students already fluent in standard English and from diverse socio-economic status and levels of math achievement. (For example, see Fuson, Atler, Roedel, & Zaccariello, 2009; Fuson & Lo Cicero, 2000; Fuson, Lo Cicero, & Smith, 1997; Lo Cicero, Fuson, & Allexaht-Snider, 1999.)

The model of balanced teaching using three phases was described earlier. This model ensures that there is a great deal of differentiation in the whole-classroom lessons for ***Math Expressions*** particularly because the teaching approaches in ***Math Expressions*** were developed and tested in a wide range of classrooms. To support the further differentiation described in the **Mastery Learning Loop,** a wide range of supports are available from which teachers can choose for the specified intervention days. For example, every lesson in ***Math Expressions*** includes intervention, on-level, and challenge differentiation to support students in their continued learning. Some other features and resources in ***Math Expressions*** that support effective differentiation and focused instruction include:

- Differentiated Instruction Activities, which appear both in the Teacher Edition and in a classroom kit.
- Leveled Math Writing Prompts, which provide opportunities for in-depth thinking and analysis.
- Support for English Language Learners, included in each lesson.
- The Math Center Challenges Easel, which includes activities, projects, and puzzlers to help the highest math achievers reach their potential.
- Expresiones en Matemáticas supports English learners with Spanish-language materials.
- The Bilingual eGlossary includes audio, graphics, and animation in both English and Spanish.

As an example, see this overview of differentiation from Grade 3 Teacher's Edition:

Differentiation and Intervention in *Math Expressions:* Individualizing Instruction Activities	
Differentiated Instruction Cards	Intervention, On Level, Challenge—in every lesson
Math Writing Prompts	Intervention, On Level, Challenge—in every lesson
Math Center Challenges	Advanced—four in every unit
English Language Learners	In every lesson
Digital Resources	• Soar to Success Math for online intervention • MegaMath for practice and application of skills and concepts • Destination Math® for problem solving practice and challenge

Differentiated Instruction

Math Expressions lessons are designed to accommodate a wide range of student learning styles and academic skills. A variety of lesson features and program resources incorporate strategies and opportunities for differentiating instruction.

English Language Learners

Present this problem to all students. Offer the different levels of support to meet students' levels of language proficiency.

Objective Review vocabulary used with unknowns.

Problem Write a word problem on the board. *Peter ate 5 grapes. Before lunch he had 12. How many grapes did Peter eat?* Model the equation $5 + \square = 12$.

BEGINNING

Say: 5 is the start addend. The total is 12 We don't know the other addend. The box is the *unknown* addend. Say: Now Peter has ___. 5 grapes The start addend is 5. We don't know how many grapes Peter ate. It is *unknown*. Before lunch Peter had ____. 12 grapes The total is ____. 12

INTERMEDIATE

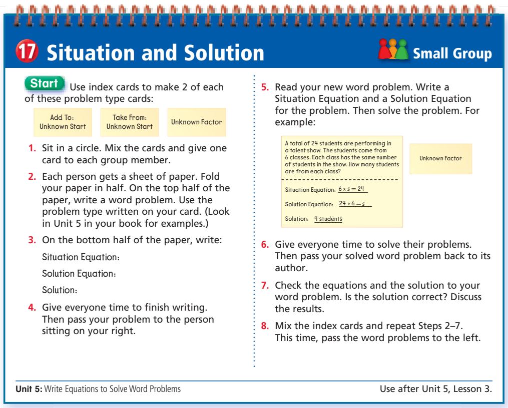
Ask: What do we know? Peter has 5 grapes now. He had 12 before lunch. What don't we know? how many grapes Peter ate

ADVANCED

Help students identify the known addend and total. Ask: Do we know the second addend? no Say: It is unknown. Ask: Do we add or subtract the unknown to find the total? add

Differentiated Instruction: Individualizing Instruction Activities	
Differentiated Instruction Cards	*Intervention • On Level • Challenge in every lesson*
Math Writing Prompts	*Intervention • On Level • Challenge in every lesson*
Math Center Challenges	*Advanced: 4 in every unit*
English Language Learners	*In every lesson*

Ready-Made Math Challenge Centers



17 Situation and Solution — Small Group

Start Use index cards to make 2 of each of these problem type cards:

Add To: Unknown Start | Take From: Unknown Start | Unknown Factor

1. Sit in a circle. Mix the cards and give one card to each group member.
2. Each person gets a sheet of paper. Fold your paper in half. On the top half of the paper, write a word problem. Use the problem type written on your card. (Look in Unit 5 in your book for examples.)
3. On the bottom half of the paper, write:

   Situation Equation:

   Solution Equation:

   Solution:
4. Give everyone time to finish writing. Then pass your problem to the person sitting on your right.
5. Read your new word problem. Write a Situation Equation and a Solution Equation for the problem. Then solve the problem. For example:

   A total of 24 students are performing in a talent show. The students come from 6 classes. Each class has the same number of students in the show. How many students are from each class?

   Situation Equation: $6 \times s = 24$

   Solution Equation: $24 \div 6 = s$

   Solution: 4 students

   Unknown Factor
6. Give everyone time to solve their problems. Then pass your solved word problem back to its author.
7. Check the equations and the solution to your word problem. Is the solution correct? Discuss the results.
8. Mix the index cards and repeat Steps 2–7. This time, pass the word problems to the left.

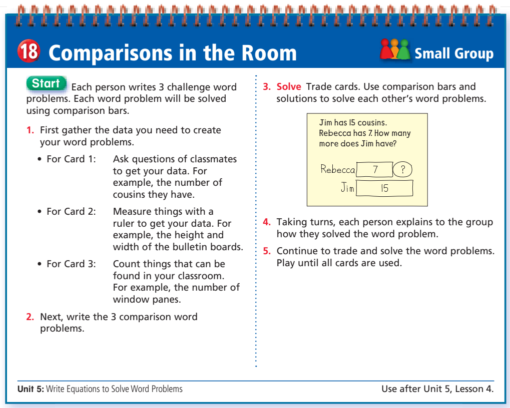
Unit 5: Write Equations to Solve Word Problems — Use after Unit 5, Lesson 3.

Grouping Small Group

Materials Index cards, sheets of paper

Objective Write and solve word problems with unknown addends and unknown factors

Common Core State Standards CC.3.NBT.2, CC.3.OA.3, CC.3.OA.4, CC.K–12.MP.2, CC.K–12.MP.7



18 Comparisons in the Room — Small Group

Start Each person writes 3 challenge word problems. Each word problem will be solved using comparison bars.

1. First gather the data you need to create your word problems.
   - For Card 1: Ask questions of classmates to get your data. For example, the number of cousins they have.
   - For Card 2: Measure things with a ruler to get your data. For example, the height and width of the bulletin boards.
   - For Card 3: Count things that can be found in your classroom. For example, the number of window panes.
2. Next, write the 3 comparison word problems.
3. **Solve** Trade cards. Use comparison bars and solutions to solve each other's word problems.

   Jim has 15 cousins. Rebecca has 7. How many more does Jim have?

   Rebecca 7 ?

   Jim 15
4. Taking turns, each person explains to the group how they solved the word problem.
5. Continue to trade and solve the word problems. Play until all cards are used.

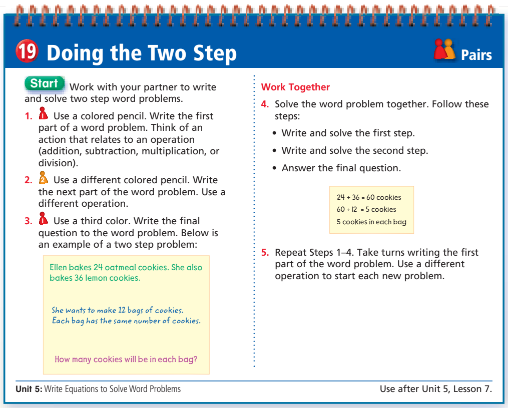
Unit 5: Write Equations to Solve Word Problems — Use after Unit 5, Lesson 4.

Grouping Small Group

Materials Heavy paper, ruler

Objective Write and solve comparison word problems

Common Core State Standards CC.3.NBT.2, CC.K–12.MP.4, CC.K–12.MP.6



19 Doing the Two Step — Pairs

Start Work with your partner to write and solve two step word problems.

1. Use a colored pencil. Write the first part of a word problem. Think of an action that relates to an operation (addition, subtraction, multiplication, or division).
2. Use a different colored pencil. Write the next part of the word problem. Use a different operation.
3. Use a third color. Write the final question to the word problem. Below is an example of a two step problem:

   Ellen bakes 24 oatmeal cookies. She also bakes 36 lemon cookies.

   She wants to make 12 bags of cookies. Each bag has the same number of cookies.

   How many cookies will be in each bag?

Work Together

4. Solve the word problem together. Follow these steps:
   - Write and solve the first step.
   - Write and solve the second step.
   - Answer the final question.

   $24 + 36 = 60$ cookies

   $60 \div 12 = 5$ cookies

   5 cookies in each bag
5. Repeat Steps 1–4. Take turns writing the first part of the word problem. Use a different operation to start each new problem.

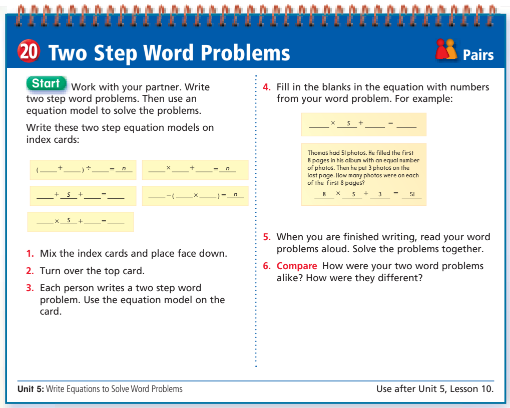
Unit 5: Write Equations to Solve Word Problems — Use after Unit 5, Lesson 7.

Grouping Pairs

Materials Colored pencils

Objective Write and solve two step problems using addition, subtraction, multiplication, and division

Common Core State Standards CC.3.OA.8, CC.K–12.MP.1, CC.K–12.MP.6



20 Two Step Word Problems — Pairs

Start Work with your partner. Write two step word problems. Then use an equation model to solve the problems.

Write these two step equation models on index cards:

$(\_\_ + \_\_) \div \_\_ = n$

$\_\_ \times \_\_ + \_\_ = n$

$\_\_ + s + \_\_ = \_\_$

$\_\_ - (\_\_ \times \_\_) = n$

$\_\_ \times s + \_\_ = \_\_$

1. Mix the index cards and place face down.
2. Turn over the top card.
3. Each person writes a two step word problem. Use the equation model on the card.
4. Fill in the blanks in the equation with numbers from your word problem. For example:

   $\_\_ \times s + \_\_ = \_\_$

   Thomas had 51 photos. He filled the first 8 pages in his album with an equal number of photos. Then he put 3 photos on the last page. How many photos were on each of the first 8 pages?

   $8 \times s + 3 = 51$
5. When you are finished writing, read your word problems aloud. Solve the problems together.
6. **Compare** How were your two word problems alike? How were they different?

Unit 5: Write Equations to Solve Word Problems — Use after Unit 5, Lesson 10.

Grouping Pairs

Materials Index cards

Objective Write and solve two step problems using addition, subtraction, multiplication, and division

Common Core State Standards CC.3.OA.8, CC.K–12.MP.1, CC.K–12.MP.3

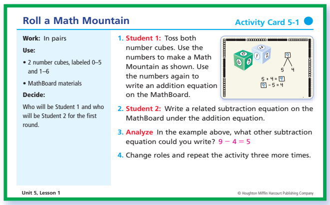
And, see this example, which follows Grade 3, Unit 1, Lesson 1 in the Teacher's Edition.

Teaching the Lesson | **Differentiated Instruction** | Homework and Spiral Review

Intervention

RtI Tier 2

for students having difficulty



Activity Notes In this activity, students toss two number cubes and use them to make two numbers. Then students make a Math Mountain and write an addition equation.

For most pairs of numbers, there are two possible addition equations and two possible subtraction equations.

Math Writing Prompt

Explain How You Know Explain how you can find an unknown number in a Math Mountain.

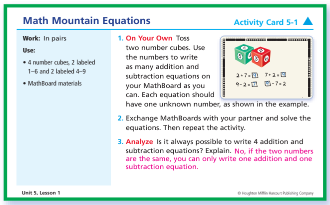
Soar to Success Math Intervention

Software Support
Warm-Up 14.57
Solve Equations with Addition and Subtraction

On Level

RtI Tier 1

for students having success



Activity Notes In this activity, students toss two number cubes and use the numbers to make as many addition and subtraction equations as they can.

Students can check their work by looking at each equation to be sure that the same two or three numbers are used.

Math Writing Prompt

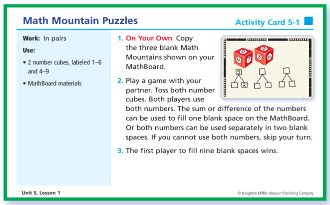
Compare and Contrast How are an Add To problem and a Take From problem different? How are they alike?

MegaMath

Software Support
Ice Station Exploration:
Arctic Algebra, Level Y

Challenge

for students seeking a challenge



Activity Notes In this activity, students fill the empty spaces on their Math Mountain using the sums or differences of numbers made from their number cubes.

Students should not reveal how they will use the two numbers after each roll. After the game ends, the winner's partner checks for errors.

Math Writing Prompt

Explain Your Thinking Joni said, "I can either add or subtract to solve a Put Together/Take Apart problem." Write an explanation of what Joni means.

Destination Math®

Software Support
Course I: Module 2: Unit 1:
Session 4: Sums within 20

***Math Expressions* supports a Response to Intervention (RtI) instructional model.** With the strong comprehensive assessment program, varied assessment tools, and opportunities for differentiation, teachers use materials, activities, and resources aligned to student needs.

Within the core program, daily opportunities for **Quick Practice,** guided practice with feedback in the **Helping Learning Community,** and regular cumulative review through **Homework and Remembering,** all work together to make Tier 1 prevention accessible to the great majority of students in the classroom.

Tier 2 students benefit from ongoing use of **Math Boards, Math Talk,** and guided practice and feedback through the **Helping Learning Community.** They may also use some of the Differentiated Intervention Cards provided with each lesson of ***Math Expressions*** on the intervention days. These are designed for use in small groups and extend research-based lesson practices to Tier 2 students.

Tier 3 students can benefit from using the same research-based lessons and conceptual supports but taking more time to do so with the support of a knowledgeable leader.

***Soar to Success* Math** provides a digital path to intervention, with flexible options for addressing Tiers 1, 2, and 3, along with an assessment, management, and reporting system to track student progress.

The program resource, ***Response to Intervention for the Common Core State Standards,*** offers focused, standards-aligned instruction through tiered teaching lessons.

Response to Intervention for the Common Core State Standards	
Tier 1	**Grade-level content:** Students have the opportunity to experience new instruction on grade-level concepts they have not yet mastered.
Tier 2	**Prerequisite skills:** Students have the chance to experience new instruction in the prerequisite skills they need for success with grade-level content.
Tier 3	**Scaffolded activities:** Students use real-world situations and carefully scaffolded examples to build the foundational knowledge they need to succeed with grade-level content. Once students achieve success at Tier 3, they can move to Tier 2, and then to Tier 1 for additional practice.

In 2009, the What Works Clearinghouse of the Institute of Education Sciences published a report on *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools.* The report recommends the following practices; the table on the next page shows how these practices are developed in ***Math Expressions.***

What Works Clearinghouse Recommendations for RtI Activities	How This is Developed in *Math Expressions*
Screen all students to identify those at risk for potential mathematics difficulties and provide interventions to students identified as at risk. (Tier 1)	***Math Expressions*** offers a comprehensive assessment program, which include diagnostic tools and ongoing formative assessment resources. For more details, see Strand 5 of this report. The **Math Talk Learning Community** also enables the teacher to continually assess students at risk and how they are progressing.
Instructional materials for students receiving interventions should focus intensely on in-depth treatment of whole numbers (–5) and rational numbers (4-8). (Tiers 2 and 3)	***Math Expressions*** aligns to the Common Core State Standards for Mathematics and follows a "Big Ideas" organizational format, in which each grade level focuses on deeply learning essential content and emphasizes depth over breadth.
Instruction during intervention should provide models, allow for verbalization of thought processes, and offer practice, feedback, and review. (Tiers 2 and 3)	**Math Drawings** provide visual models. **Math Boards** and **Math Talk** provide the chance for students to share strategies and methods. The ***Math Expressions*** Helpful Learning Community offers guided practice and feedback.
Interventions should include instruction on solving word problems. (Tiers 2 and 3)	**Word Problems** are included throughout ***Math Expressions,*** across grade levels, and include problems involving addition, subtraction, multiplication, division, graph problems, two-step word problems, and many more.
Intervention materials should include opportunities to work with visual representations. (Tiers 2 and 3)	Throughout ***Math Expressions***, the use of **Math Drawings** give teachers and students the chance to use visuals for math teaching and learning.
Intervention sessions should devote about 10 minutes to fluency. (Tiers 2 and 3)	Every lesson in ***Math Expressions*** opens with Quick Practice.
Students' progress should be monitored. (Tiers 2 and 3)	***Math Expressions*** offers a comprehensive assessment program including continual formative assessment by the teacher. For more details, see Strand 5 of this report.
Intervention should be designed to motivate students. (Tiers 2 and 3)	To be motivated to learn, students must have the expectation that they can learn and a belief that their learning has a value (National Research Council, 2001). Scaffolding and carefully sequence learning activities mean that students feel confident in their abilities to learn. Real-world activities in context make learning meaningful—and therefore, motivating.

STUDENT MATH ACHIEVEMENT IN ELEMENTARY SCHOOL: A COMPARISON OF FOUR MATH CURRICULA

Research Design

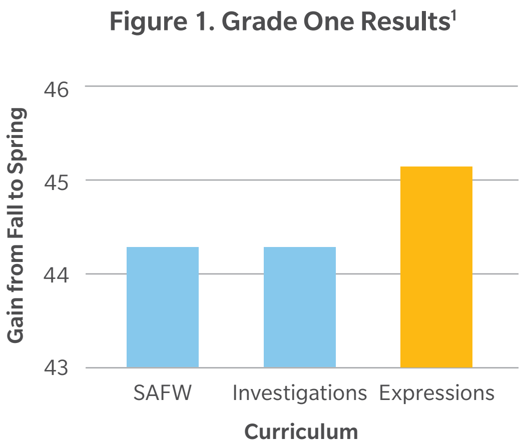
Researchers from Mathematic Policy Institute used an experimental research design, a Randomized Control Trial (RCT), to determine the effectiveness of four elementary math curricula on student academic achievement. The study compared math achievement among students using either, ***Math Expressions, Saxon Math, Scott Foresman-Addison Wesley,*** or ***Investigations in Numbers, Data, and Space.*** This research-design is considered to be a "gold-standard" design and is one of only two experimental research designs that meet the What Works Clearinghouse research standards.

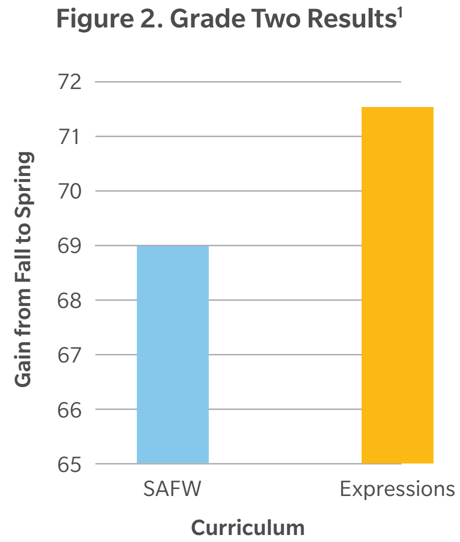
Study Procedures

Twelve, geographical disperse districts throughout the country were included in the study. Schools were randomly assigned to provide instruction using one of the four identified curricula during the 2007-2008 school year. Student math achievement was measured during the fall and spring of the year using the math assessment designed for the Early Childhood Study-Kindergarten Class of 1998–99 (ECLS-K). Student progress, from fall 2007 to spring 2008, was examined for 4,716 first grade students attending 109 schools and 3,344 second grade students attending 71 schools. Hierarchical linear modeling (HLM) assessed the effect of using the various curricula on student math achievement.

Study Findings

At first grade, students using ***Math Expressions*** (Expressions) had significantly higher math achievement when compared to students using SFAW and Investigations; there was no significant difference between ***Math Expressions*** students and students using ***Saxon Math*** (Figure 1). These results are indicated that on average, students using ***Math Expressions*** were performing four percentile points higher than students using ***Scott Foresman-Addison Wesley*** or ***Investigations.*** At second grade, ***Math Expressions*** students outperformed SFAW students; on average, these students were performing fiver percentile points higher than students using ***Scott Foresman-Addison Wesley.*** There were no other significant differences among second grade students (Figure 2). These results occurred even though ***Saxon Math*** teachers reported an average of one hour more math instruction, per week, than teachers using the other three programs.





[1]Adjusted mean gains can be computed from results presented on pp. D15–D18

Conclusions

According to the authors, the results of the study indicate that at both the first and second grade the "curriculum mattered." ***Math Expression*** students were performing at higher levels of math achievement when compared to students using other programs (p. 77). These results are dramatic given the rigorous nature of the research design and statistical analysis used to examine the effects of these programs. The findings of this experimental evaluation provide strong evidence that ***Math Expressions*** is an effective approach to mathematics instruction at the elementary school level.

Reference: Agondi, R., Harris, B., Thomas, M., Murphy, R., & Gallagher, L. (2010). Achievement Effects of Four Early Elementary School Math Curricula: Findings for First and Second Graders (NCEE 2011–4001). Washington, DC: U.S. Department of Education

The randomized control trial (RCT) research designed utilized in this study is the strongest research design for determining causal effects in education research (see Shadish, Cook, & Campbell, 2002) and is one of two research designs that meet the evidence standards for What Works Clearinghouse. See the charts on the following pages

IMPROVED ACHIEVEMENT RESULTS FROM SCHOOL DISTRICTS USING *MATH EXPRESSIONS COMMON CORE*

Edgar SD (WI)

District Demographics

- Rural district with 1 elementary school and 2 secondary schools with a current enrollment of over 640 students in Grades PK–12.

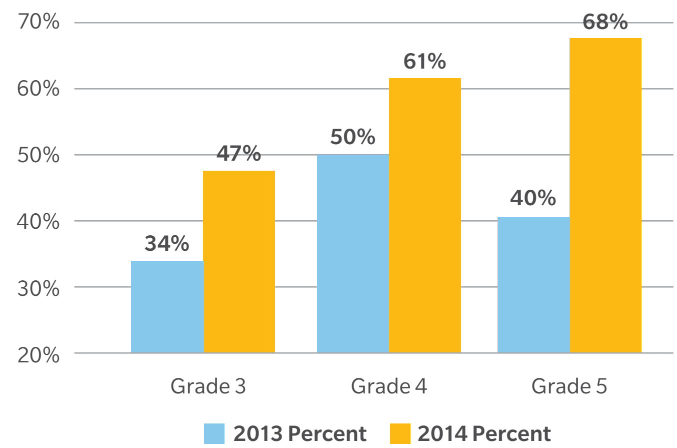
Student Ethnicities

- Caucasian 97%
- Hispanic 1%
- Black 1%
- Asian 1%
- 11% Students have an Individualized Education Program (IEP).

Measure: Wisconsin Knowledge and Concepts Examinations (WKCE)

Period of Evaluation: 2013 (Baseline) to 2014

Edgar SD Grades 3, 4, and 5 WKCE
Percent of Students **Proficient and Advanced**



Source of assessment data: http://data.dpi.state.wi.us:80/
Source of demographic data: MDR

Lancaster SD (NY)

District Demographics

- Suburban district with 4 elementary schools and 3 secondary schools with a current enrollment of over 5800 students in Grades K–12.

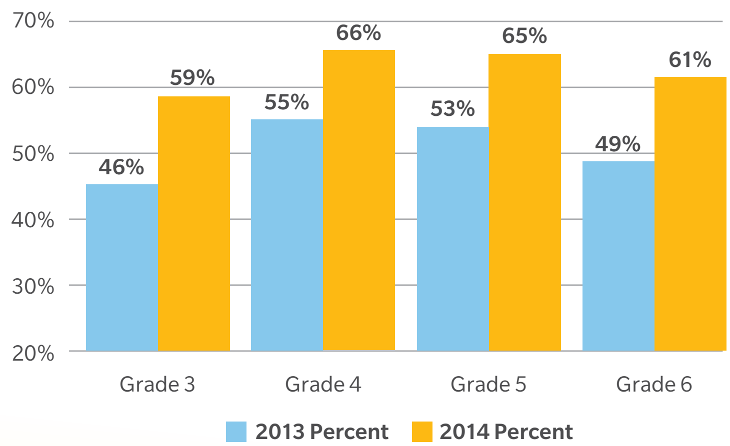
Student Ethnicities

- Caucasian 94%
- Hispanic 2%
- Other 4%
- 15% Students have an Individualized Education Program (IEP).

Measure: New York State Test (NYST)

Period of Evaluation: 2013 (Baseline) to 2014

Lancaster SD Grades 3, 4, 5, and 6 NYST
Percent of Students **Proficient and Advanced**



Source of assessment data: http://www.p12.nysed.gov/irs/ela-math/
Source of demographic data: MDR

Ripon SD (WI)

District Demographics

- Suburban district with 3 elementary schools and 5 secondary schools with a current enrollment of over 1700 students in Grades PK–12.

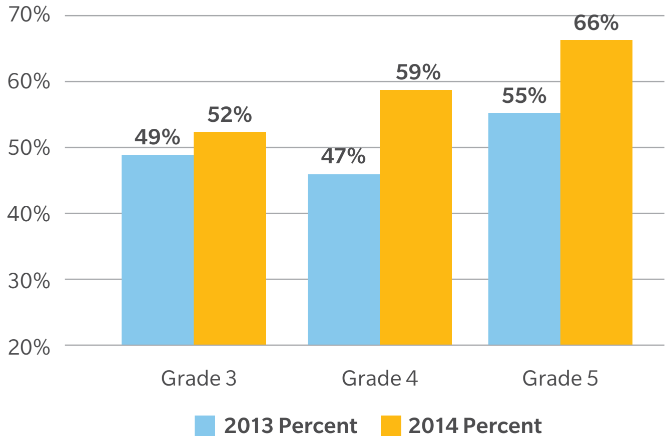
Student Ethnicities

- Caucasian 90%
- Hispanic 7%
- Other 3%
- 10% Students have an Individualized Education Program (IEP).

Measure: Wisconsin Knowledge and Concepts Examinations (WKCE)

Period of Evaluation: 2013 (Baseline) to 2014

Ripon Area SD Grades 3, 4, and 5 WKCE Percent of Students Proficient and Advanced



Source of assessment data: http://data.dpi.state.wi.us:80/
Source of demographic data: MDR

Swartz Creek SD (MI)

District Demographics

- Suburban district with 3 elementary schools and 5 secondary schools with a current enrollment of over 1700 students in Grades K–12.

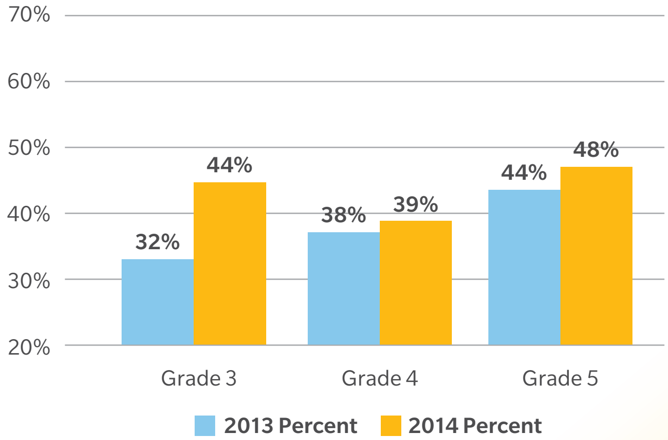
Student Ethnicities

- Caucasian 91%
- Black 5%
- Hispanic 2%
- Indian 1%
- Asian 1%
- 13% Students have an Individualized Education Program (IEP).

Measure: Michigan Educational Assessment Program (MEAP)

Period of Evaluation: 2013 (Baseline) to 2014

Swartz Creek SD Grades 3, 4, and 5 MEAP Percent of Students Proficient and Advanced



Source of assessment data: https://www.mischooldata.org/DistrictSchoolProfiles/AssessmentResults/Meap/MeapPerformanceSummary.aspx
Source of demographic data: MDR

Mohawk Train SD (MA)

District Demographics

- Rural district with 4 elementary schools and 1 secondary school. The district has current enrollment of over 1000 students in Grades PK–12.

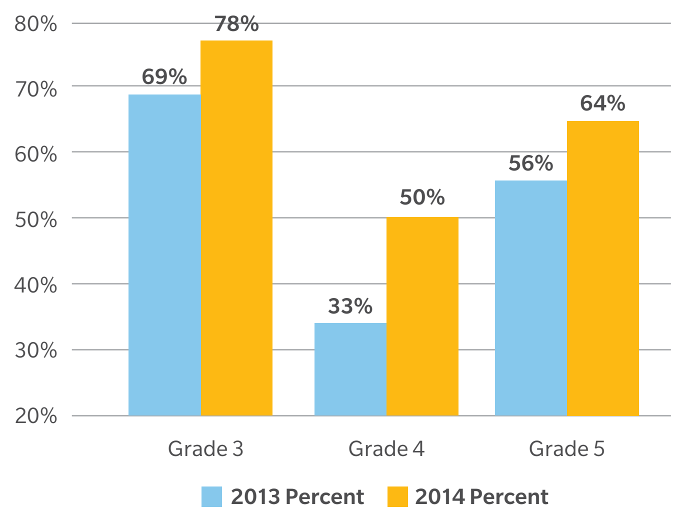
Student Ethnicities

- Caucasian 92%
- Hispanic 4%
- Other 4%
- 120% Students have an Individualized Education Program (IEP).

Measure: Massachusetts Comprehensive Assessment System (MCAS)

Period of Evaluation: 2013 (Baseline) to 2014

Mohawk Trail SD Grades 3, 4, and 5 MCAS
Percent of Students **Proficient and Advanced**



Source of assessment data: http://www.doe.mass.edu/mcas/
Source of demographic data: MDR

Orchard Park SD (NY)

District Demographics

- Suburban district with 4 elementary schools and 2 secondary with a current enrollment of over 4900 students in Grades PK–12.

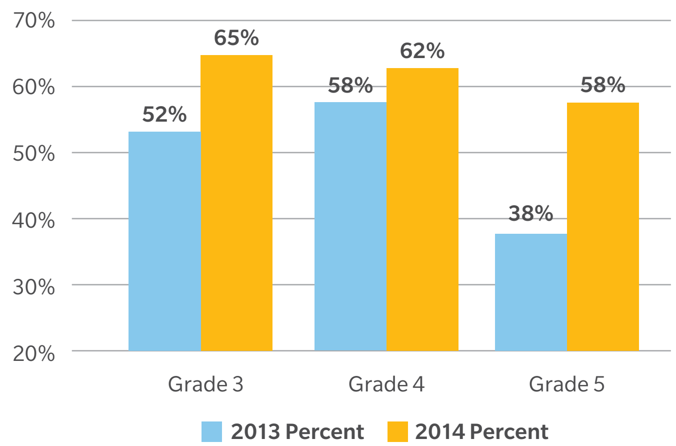
Student Ethnicities

- Caucasian 94%
- Hispanic 2%
- Asian 2%
- Other 4%
- 14% Students have an Individualized Education Program (IEP).

Measure: New York State Test (NYST)

Period of Evaluation: 2013 (Baseline) to 2014

Orchard Park SD Grades 3, 4, and 5 NYST
Percent of Students **Proficient and Advanced**



Source of assessment data: http://www.p12.nysed.gov/irs/ela-math/
Source of demographic data: MDR

West Irondequoit SD (NY)

District Demographics

- Suburban district with 6 elementary schools and 4 secondary with a current enrollment of over 3600 students in Grades K–12.

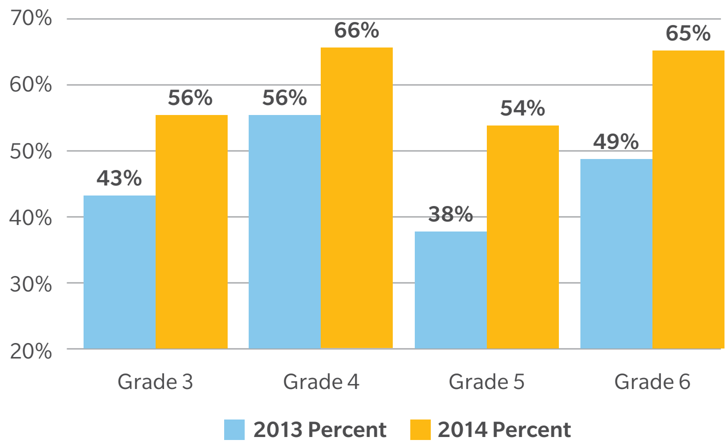
Student Ethnicities

- Caucasian 76%
- Hispanic 10%
- Black 9%
- Other 5%
- 14% Students have an Individualized Education Program (IEP).

Measure: New York State Test (NYST)

Period of Evaluation: 2013 (Baseline) to 2014

West Irondequoit SD Grades 3, 4, 5, and 6 NYST
Percent of Students **Proficient and Advanced**



Source of assessment data: http://www.p12.nysed.gov/irs/ela-math/
Source of demographic data: MDR

SUMMARY OF FULL YEAR STUDY OF *MATH EXPRESSIONS* COMMON CORE

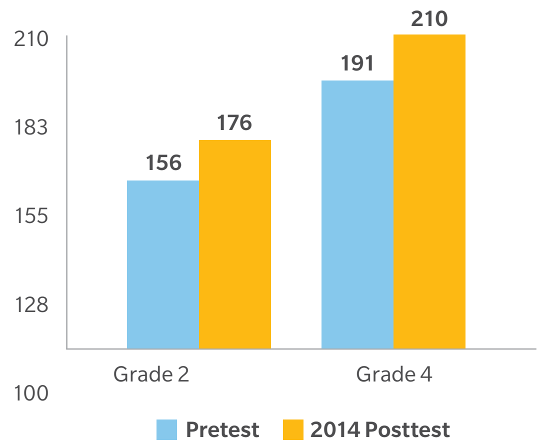
Houghton Mifflin Harcourt commissioned Educational Research Institute of America (ERIA) to conduct a yearlong study of the effectiveness of ***Math Expressions Common Core*** during the 2013–2014 school year1.

The study was conducted at Grades 2 and 4 among a total of 54 teachers (30 Grade 2 teachers and 24 Grade 4 teachers) from eight schools in four states (AZ, IL, MI, WI). The final analytic sample consisted of 579 Grade 2 students and 503 Grade 4 students for which researchers had pretest and posttest scores. The Iowa Test of Basic Skills (ITBS) was utilized to measure student math achievement over the course of the study-level 8 of the test was administered to Grade 2 students and level 10 was used at Grade 4. All students were administered the ITBS twice, once as a pretest in September of 2013 and again as a posttest in May 2014.

Statistical analysis of student standard scores indicated that at both grade levels, students made statistically significant ($p < .05$) progress on the ITBS over the course of the academic year. In addition to this significant growth, the percentage of students scoring above the expected grade equivalence levels increased substantially over the course of the study.

Based on these findings, researchers from ERIA concluded that ***Math Expressions Common Core*** was effective in improving the mathematics knowledge, skills, and understanding of students in Grades 2 and 4.

Figure 1
Grades 2 and 4 Students
Iowa Test of Basic Skill Standard
Gains from Pretest to Posttest



[1] Educational Research Institute of America (July, 2014). A study of the instructional effectiveness of ***Math Expressions Common Core*** 2013 ©: Report 473. Bloomington, IN: Author.

PROJECT REFERENCES AND ADDITIONAL RESEARCH SUPPORT FOR *MATH EXPRESSIONS:* ORGANIZED BY MATHEMATICAL FOCUS

Math Expressions is based on the research results of the National Science Foundation (NSF) funded Children's Math Worlds (CMW) research project, and is the only U.S. curriculum developed using the methods of learning science design research. ***Math Expressions*** author Dr. Karen Fuson's work leading the CMW research project was foundational in identifying the key components for successful mathematics learning, the focus and sequence of content, and the effective instructional practices built into ***Math Expressions***.

As part of the CMW research project, work was done in classrooms and in interviews for at least four to five years on major topics at each grade level, with continual revision of the teaching and learning materials. The goal was to identify supported learning paths through major math domains that could be coherently woven across grades. The research tasks included:

- Identifying typical student errors and how to overcome them
- Developing accessible and mathematically-desirable algorithms that relate to common algorithms but that all students can understand and explain
- Choosing math drawings that facilitate understanding of the domain situations or quantities

Listed below are papers reporting research on which ***Math Expressions*** is based. The following papers describe (1) research on teaching and learning that provides part of the research base from which the Children's Math Worlds Research Project (CMW)* was developed and (2) research reports that document the individual design studies and their success with students. More research papers exist in draft and research summary form based on the intensive ten-year period of progressive refinement of the curriculum using extensive observations and feedback from teachers, and more papers will be written to summarize research results in other areas.

Webcasts are also available that describe the research-based learning paths and visual supports used in ***Math Expressions*** for major math domains. Contact your HMH® Account Executive for details.

Research Concerning Broad Aspects of the *Math Expressions* Classroom

*These articles describe central aspects of **Math Expressions** classrooms in action to create communication, confidence, and competence.*

Hufferd-Ackles, K., Fuson, K.C., & Sherin, M.G. (2004). Describing levels and components of a Math-Talk Learning Community. *Journal for Research in Mathematics Education,* 35(2), 81–116.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2015). Describing levels and components of a Math-Talk Learning Community. In E. A. Silver & P. A. Kenney (Eds.), *More lessons learned from research: Volume 1: Useful and usable research related to core mathematical practices,* (pp. 125–134). Reston, VA: NCTM.

Fuson, K.C., De La Cruz, Y., Smith, S., Lo Cicero, A., Hudson, K., Ron, P., & Steeby, R. (2000). Blending the best of the 20th century to achieve a mathematics equity pedagogy in the 21st century. In M.J. Burke & F.R. Curcio (Eds.), Learning mathematics for a new century, (pp. 197–212). Reston, VA: National Council of Teachers of Mathematics.

Fuson, K.C., Atler, T., Roedel, S., & Zaccariello, J. (2009). Building a nurturing, visual, Math-Talk teaching-learning community to support learning by English Language Learners and students from backgrounds of poverty. New England Mathematics Journal, XLI(May), 6–16.

Research on the Learning Path Approach in the *Math Expressions* Classroom

These articles describe the balanced learning path teaching approach used in **Math Expressions** *classrooms.*

Fuson, K.C., Murata, A., & Abrahamson, D. (2014). Using learning path research to balance mathematics education: Teaching/learning for understanding and fluency. In R. Cohen Kadosh & A. Dowker (Eds.), *Oxford handbook of numerical cognition. Oxford Handbooks Online* (pp. 1036–1054). Oxford, England: Oxford University Press.

Murata, A., & Fuson, K.C. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using ten. *Journal for Research in Mathematics Education,* 37(5), 421–456.

Fuson, K.C. & Murata, A. (2007). Integrating NRC principles and the NCTM Process Standards to form a Class Learning Path Model that individualizes within whole-class activities. National Council of Supervisors of Mathematics. *Journal of Mathematics Education Leadership,* 10(1), 72–91.

Fuson, K.C. (2009). Avoiding misinterpretations of Piaget and Vygotsky: Mathematical teaching without learning, learning without teaching, or helpful learning-path teaching? *Cognitive Development,* 24(4), 343–361.

Overviews of Research on Computation and Word Problems

These articles summarize effective ways to teach computation and word problems.

Fuson, K.C. (2003). Developing mathematical power in whole number operations. In J. Kilpatrick, W.G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 68–94). Reston, VA: National Council of Teachers of Mathematics.

Fuson, K.C. (1992). Research on whole number addition and subtraction. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243–275). New York: Macmillan.

Research on Using *Math Expressions* Approaches and Activities in Diverse, Urban Schools

These articles describe central aspects of developing a **Math Expressions** *classroom in urban Latino classrooms, where CMW started. (Later research tested the curricular approaches in a wide range of schools, including advantaged schools with many high-achieving students, but the initial research was conducted in urban schools with diverse populations, including many underrepresented minorities and English learners.)*

Fuson, K.C., Lo Cicero, A., Hudson, K., & Smith, S.T. (1997). Snapshots across two years in the life of an Urban Latino Classroom. In J. Hiebert, T. Carpenter, E. Fennema, K.C. Fuson, D. Wearne, H. Murray, A. Olivier, & P. Human. *Making sense: teaching and learning mathematics with understanding* (pp. 129–159). Portsmouth, NH: Heinemann.

Lo Cicero, A., Fuson. K.C., & Allexaht-Snider, M. (1999). Making a difference in Latino children's math learning: Listening to children, mathematizing their stories, and supporting parents to help children. In L. Ortiz-Franco, N.G. Hernandez, & Y. De La Cruz (Eds.), *Changing the faces of mathematics: Perspectives on Latinos* (pp. 59–70). Reston, Virginia: National Council of Teachers of Mathematics.

Fuson, K.C., & Lo Cicero, A. (2000). El Mercado in Latino primary math classrooms. In M.L. Fernandez (Ed.), *Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, p. 453). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Research on *Math Expressions* Approaches in Kindergarten

These articles describe the ways that **Math Expressions** *approaches in Kindergarten support ambitious, international levels of conceptions of teen numbers as tens and ones, as well as embedded numbers that can support advanced addition and subtraction methods.*

Ho, C.S., & Fuson, K.C. (1998). Children's knowledge of teens quantities as tens and ones: Comparisons of Chinese, British, and American kindergartners. *Journal of Educational Psychology,* 90(3), 536–544.

Fuson, K.C., Grandau, L., & Sugiyama, P.A. (2001). Achievable numerical understandings for all young children. Invited paper for the "Research into Practice" series. *Teaching Children Mathematics,* 7(9), 522–526.

Other papers are being prepared for publication.

Research on Single-Digit Addition and Subtraction

These articles describe the learning path in single-digit addition and subtraction moving to powerful and general methods.

Fuson, K.C., Perry, T., & Kwon, Y. (1994). Latino, Anglo, and Korean children's finger addition methods. In J.E.H. van Luit (Ed.), Research on learning and instruction of mathematics in kindergarten and primary school (pp. 220–228). Doetinchem/Rapallo, The Netherlands: Graviant.

Fuson, K.C., & Secada, W.G. (1986). Teaching children to add by counting on with finger patterns. Cognition and Instruction, 3(3), 229–260.

Fuson, K.C. (1986). Teaching children to subtract by counting up. Journal for Research in Mathematics Education, 17(3), 172-189. (This paper was chosen as the best research article of 1986 by the Research Advisory Council of the National Council of Teachers of Mathematics.)

Fuson, K.C., & Kwon, Y. (1992). Korean children's single-digit addition and subtraction: Numbers structured by ten. Journal for Research in Mathematics Education, 23(2), 148–165.

Duncan, A., Lee, H., & Fuson, K.C. (2000). Pathways to early number concepts: Use of 5- and 10-structured representations in Japan, Taiwan, and the United States. In M.L. Fernandez (Ed.), Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2, p. 452). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Fuson, K.C. (1987). Adding by counting on with one-handed finger patterns. The Arithmetic Teacher, 35(1), 38–41. (Invited paper; first article in the new "Research into Practice" series.)

Fuson, K.C. (1988). Subtracting by counting up with one-handed finger patterns. The Arithmetic Teacher, 35(5), 29–31. (Invited paper for the "Research into Practice" series.)

Murata, A., & Fuson, K.C. (2001). Learning paths to 5- and 10-structured understanding of quantity: Addition and subtraction solution strategies of Japanese children. In R. Speiser, C.S. Maher, & C. Walter (Eds.) Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 639–646). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Murata, A., & Fuson, K.C. (2006). Teaching as assisting individual constructive paths within an interdependent class learning zone: Japanese first graders learning to add using ten. Journal for Research in Mathematics Education, 37(5), 421–456.

Research on Multi-Digit Addition and Subtraction

These articles describe the learning path in multi-digit addition and subtraction moving to powerful and general methods.

Fuson, K.C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. Cognition and Instruction, 7(4), 343–403.

Fuson, K.C. (1990). Issues in place-value and multidigit addition and subtraction learning and teaching. Journal for Research in Mathematics Education, 21(4), 273–280.

Fuson, K. C. & Beckmann, S. (Fall/Winter, 2012-2013). Standard algorithms in the Common Core State Standards. National Council of Supervisors of Mathematics: Journal of Mathematics Education Leadership, 14(2), 14–30.

Fuson, K.C., & Burghardt, B.H. (2003). Multi-digit addition and subtraction methods invented in small groups and teacher support of problem solving and reflection. In A.J. Baroody & A. Dowker (Eds.), The development of arithmetic concepts and skills: Constructing adaptive expertise (pp. 267–304). Mahwah, NJ: Lawrence Erlbaum Associates.

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Fuson, K.C., & Smith, S.T. (1997). Supporting multiple 2-digit conceptual structures and calculation methods in the classroom: Issues of conceptual supports, instructional design, and language. In M. Beishuizen, K.P.E. Gravemeijer, & E.C.D.M. van Lieshout (Eds.), The role of contexts and models in the development of mathematical strategies and procedures (pp. 163–198). Utrecht, The Netherlands: CD-B Press/The Freudenthal Institute.

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Fuson, K.C., Wearne, D., Hiebert, J., Human, H., Murray, A., Olivier, A., Carpenter, T., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. Journal for Research in Mathematics Education, 28(2), 130–162.

Research on Word Problems

These articles describe the learning path in teaching and learning the full range of word problems.

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Fuson, K.C., & Willis, G.B. (1989). Second graders' use of schematic drawings in solving addition and subtraction word problems. Journal of Educational Psychology, 81(4), 514–520.

Lo Cicero, A., De La Cruz, Y., & Fuson, K.C. (1999). Teaching and learning creatively: Using children's narratives. Teaching Children Mathematics, 5(9), 544–547.

Stigler, J., Fuson, K.C., Ham, M., & Kim, M.S. (1986). An analysis of addition and subtraction word problems in Soviet and American elementary textbooks. Cognition and Instruction, 3(3), 153–171.

Research on Single-Digit and Multi-Digit Multiplication and Division

These articles describe the learning path in single-digit and multi-digit multiplication and division leading to powerful and general methods.

Fuson, K. C. & Beckmann, S. (Fall/Winter, 2012-2013). Standard algorithms in the Common Core State Standards. National Council of Supervisors of Mathematics Journal of Mathematics Education Leadership, 14(2), 14–30.

Izsák, A., & Fuson, K.C. (2000). Students' understanding and use of multiple representations while learning two-digit multiplication. In M.L. Fernandez (Ed.), Proceedings of the Twenty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 714–721). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

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Notes

Notes

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